Imperial College London

Quest for Time in Unimodular Gravity Theories

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Date: September 22th, 2023

Submitted in partial fulfillment of the requirements
for the Degree of Master of Science
in Quantum Fields and Fundamental Forces

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Abstract:

In this thesis, we study time through the perspective of unimodular gravity. For this, we provide a detailed review of unimodular gravity and its variants. We then introduce concepts from knot theory, Chern-Simons theories, and magnetic helicity, and use them to construct a framework within unimodular gravity theories. The result is a reinterpretation of unimodular time in terms of magnetic helicity which gives a new possible explanation for the flow of time in terms of magnetic vortices, flux tubes, and knots. The resulting theory is topological. We then expand on the Page-Wootters quantum mechanics and formulate a relativistic extension via POVMs. This introduces appropriate time observables and a factorization of Hilbert spaces that contain a system and reference frame factors. We recover standard relativistic quantum physics results and give physical meaning to the construction.

Dedication

This thesis is dedicated to my mother and father, two of the dearest persons in my life who cannot be replaced.

Declaration

The research described in this dissertation is based on work carried out at the Department of Physics, Imperial College London. This is the author's original work and is the result of collaborations with Prof. Jõao Magueijo and Farbod Sayyed Rassouli. The results of Chapter 3 will appear in the paper [22].

Acknowledgements

I would like to start by thanking Prof. Jõao Magueijo for offering me this amazing opportunity to work under his guidance on this project. His insight on questioning the most unlikely, pushing ideas to their limits, and creating the perfect working atmosphere are what kept me going throughout this dissertation. I am forever grateful to have been working on original thoughts with him.

With all the ups and downs of this year, I have made many close friends. I would like to start with Farbod Sayyed Rassouli. I am thankful to him for always being there, and with whom I was able to communicate about anything without strings attached. I am also very excited about our collaboration under Prof. Jõao Magueijo, which we always wanted to achieve together. Our journey started the very first day of the year in the common computer room where we exchanged one of my strangest conversations. This has led us to start working on projects with Prof. Amihay Hanany, with whom Farbod and I have exchanged numerous lengthy conversations and for always having his door open regarding any matter. Our friendship and collaboration will probably be a lifelong journey and I am excited that we both are moving toward our Ph.D.'s in theoretical physics.

Next, I would like to thank my dear friend Paolo Bassani, who has had a special place in my life since our General Relativity and Cosmology classes at University College London. Our lengthy conversations while we take long walks about physics and life have always been a pleasure. Our trips to Milan this year have been one of my best fun experiences to this day, and would like to repeat this! I am also glad that we did some amazing work together back in the day, and hope to continue this collaboration as well in the future with him!

I am glad to have met Greg, Giacomo, Frank, David, Coen, and Lorenzo on the way, with some back from UCL and some from Imperial. And I am not forgetting you, Santiago, for always making me and my friends smile!

Finally, I would like to thank Prof. Kellogg Stelle for joining us at the Union Bar on Friday nights to tell us about his amazing stories about physics and life's journey! His open-door policy at his office and his always answering my questions shaped my opinions about theoretical physics. Next, I would like to thank Dr. Karapet Mkrtchyan for always providing us with answers to our questions with Farbod. I am eternally grateful for Max Hutt's brilliant lectures on topological field theory that helped with my research for this dissertation. I would like to express my gratitude to Prof. Claudia de Rham for being my personal tutor throughout the year and providing me with the necessary help and support during my Ph.D. applications and other academic matters. Final words should be placed to Prof. Edward Witten, with whom we corresponded on his paper on Quantum field theory and Jones polynomials.

Contents

Chapter 1

Introduction

Gravity has many issues concerning how it works within classical and quantum frameworks. The very interesting aspect of gravity is how related it is to gauge theories, yet it is uniquely problematic by itself and thus does not fit nicely within quantum field theory and the standard model of particle physics. In this introductory chapter, we will go through some of the main issues regarding gravitational theories and will be mainly concerned with the problem of time and cosmological constant problems.

1.1 Gravity and Time

Let us first address the problem of time in gravitational theories, as this is kind of a serious and very well-known one. This happens when we wish to unify general relativity, or any other alternative classical field theory for gravity, with quantum theory. Many such attempts have been made in the past and none in the eyes of the author are fully satisfactory theories describing Nature at its most fundamental level, but merely very good theoretical and mathematical models.

Such attempts have been made through two main separate approaches: particle physicists' vs. general relativists' perspectives are the ones that dominate the literature. The particle physicist's favorite theory is no doubt string theory and its variants such as M-theory, holography, and many more. These theories all have one aspect in common and it is the fact that the background spacetime itself is kept classical, while the particles and fields that exist within such an arena are quantized using different techniques from quantum field theory. The resulting quantum theory implies a non-canonical quantization of the background spacetime and even predicts various extra space dimensions. Such theories usually come with a framework that unifies gravity with other gauge fields, with the problem of being usually too complicated or having too many symmetries to describe reality coherently. On the other hand, relativists' approach is to just take the background spacetime and canonically or covariantly quantize it. The resulting theories are usually not unified field theories, but a full quantization of spacetime, i.e. a much more desired and possibly realistic and minimalist way to obtain a theory of quantum gravity. Again, where are the other gauge fields and how they would fit within such a framework is among one of the problems with such a pathway.

The main problem with both approaches is predictability and how time is treated. Let us step back for a second to how time appears in general relativity. Einstein's theory is diffeomorphism invariant: equations must be covariant i.e. invariant under any possible set of spacetime coordinate transformations. This simply means that the laws of physics do not care about the reference frame picked, as there is no preferred frame to choose. Now within quantum physics, time enters à la Newtonian, which means it is absolute. This makes time a parameter of evolution in the equations of quantum mechanics and quantum field theory. Hence, time cannot be made a physical observable like position and momentum. We thus cannot construct a time operator. There also exists mathematical results by Pauli stating that given a self-adjoint Hamiltonian that is bounded from below, there does not exist a self-adjoint time operator canonically conjugate to the Hamiltonian operator, and this is a no-go theorem. On the other hand, Einstein's gravity has the biggest problem of being timeless when considered in its Hamiltonian ADM formulation: the Hamiltonian is a constraint and thus when quantized using the second-quantization technique, it gives frozen states, i.e. quantum states or wavefunctionals do not evolve with some time parameter since there is no such time parameter. Such issues when considered for a quantum theory of gravity make it super hard to understand how time appears within our universe, and why we do experience the passage of time or observe change around us. Thus, how can we introduce the notion of time in a quantum theory of gravity, or more generally how time should be thought of? A parameter or an observable?

The answer to this question is believed to be a mere change of mathematical framework combined with correct physical intuition. There are currently three main approaches within the literature regarding choosing a correct way for interpreting time and its fundamental role in physics and gravity: time can enter before or after quantization, or simply it has no role within a theory, but merely is an emergent concept. Throughout this thesis, we are supporting the first and second approaches, namely appropriately introducing time as an observable before or after quantization using appropriate mathematics and also having good physical reasons why one should consider such a path. Within these approaches, one usually introduces constraints by extending the phase space and then quantizing accordingly, although the latter procedure should be carried out very delicately, as will be explored in detail in this thesis.

1.2 Gravity, Fundamental Constants, and the Cosmological Constant

Unlike mathematics, physical theories are constructed by appropriately identifying fundamental constants of Nature. For instance, for relativistic gravitation, the most important constants are Newton's gravitational constant G and the speed of light c. Quantum theory is constructed around the Planck's constant $\hbar$. It is the constants that make physics describe Nature and distinguish it from pure mathematics. The fundamental issue that arises within the realm of constants is the values they take [6]. Regardless of how precise they can be measured, we still have no scientific explanation for their specific numerical values. We believe this can be obtained with physical intuition combined with the right mathematics, and we shall explore this in this thesis in detail, at least within gravitation.

Among all constants, the cosmological constant problem stands out. Note that it would be much more appropriate to mention the fact that there is no single problem with the cosmological constant, but many do exist. Let us review some of them.

The main view is the predicted value for the cosmological constant from vacuum energy contributions in quantum field theory: this is at least sixty orders of magnitude bigger than the observed value for this constant [67]. The cosmological constant thus needs to be fine-tuned by at least sixty decimal places: the fact that the total vacuum energy density in general relativity due to observations are bounded $\rho_{\mathrm{T}} \leq (\mathrm{MeV})^4$, where $\rho_{\mathrm{T}} = \rho_{\mathrm{vac}} + \frac{\Lambda}{8\pi G}$, with the QFT vacuum given by $\rho_{\mathrm{vac}} \geq (\mathrm{TeV})^4$ creates this major order of magnitude issue.

Another problem is the way Λ and G scales:

$$G\Lambda \sim 10^{-120}.$$

The smallness of the cosmological constant relative to the gravitational constant is ridiculous. Why is Λ non-zero, yet so tiny? The question can also be reversed: why is G small? Both these questions require an ability to select values on these constants from some ensemble. Thus, it makes sense to look for simple theories where G would appear as an integration constant or a global degree of freedom [35].

The real issue happens when one performs the renormalization of the vacuum energy via perturbation theory. The amount of adjustment one has applied to Λ is independent of the number of loop contributions in the computations. The bare cosmological constant needs to be tuned with extreme precision at each order in perturbation theory. Therefore, tuning once is not enough, as say one tune to first-loop, but all higher-loops are unstable, and this keeps going. This is a crazy problem that needs to be resolved in order to make sense of the vacuum.

1.3 What does Unimodular Gravity Bring and Outline

The answer to this question is that unimodular gravity theories not only provide a possible answer to the problem of time but also use constants of Nature to address this issue. Now it does not solve the problem of why the constants take such numerical values fully, nor fix the radiative instability of the cosmological constant, but provides a possible gateway.

The thesis is organized as follows: in Chapter 2, we will give a review of unimodular gravity theories with emphasis on how they resolve the problem of time and some aspects of the constants of Nature. In Chapter 3, we will look at knot theory, topological quantum field theories such as the Chern-Simons theory in $2+1$-dimensions, and magnetic helicity that appears within magneto-hydrodynamic theory. The results established in this chapter will be useful for unimodular time and how even time might have topological properties. In fact, we will see that spacetime creation is associated with the changes in the configuration of how magnetic vortices link and knot with each other throughout constant-time hypersurfaces. Finally, in Chapter 4, we will take a step back and formulate a general framework to address the problem of time via extending the Page-Wootters formulation by including relativity, and thus possibly giving a clear mathematical meaning when quantizing unimodular gravity canonically. We also provide a conclusion and discussion chapter, laying out the main results of this dissertation.

Chapter 2

Review of Unimodular Gravity

2.1 Attempts for Unimodularity

There exist many attempts at formulating a theory of gravity that breaks partially the full diffeomorphism group of spacetime transformations. This has various motivations. One of them is due to Einstein. Another one is to explicitly break this large diffeomorphism group to only account for transverse modes. Both have their advantages and disadvantages and on-shell are equivalent theories. Let us first start by looking at Einstein's original attempt at unimodularity, followed by the true unimodular gravity theory.

2.1.1 Einstein's Original Try and Unimodular gravity

The origins of this theory go back to Einstein himself in [20] who used the unimodular condition $\sqrt{-g} = 1$ as a partial gauge fixing of diffeomorphism invariance to simplify computations in general relativity. Consider a coordinate transformation $x^\mu \mapsto \overline{x}^{\overline{\mu}}$ on a vector field V^μ. The latter transforms as a contravariant tensor of rank-1, i.e.

$$V^\mu \longmapsto \overline{V}^{\overline{\mu}} := \frac{\partial \overline{x}^{\overline{\mu}}}{\partial x^\mu} V^\mu. \tag{2.1}$$

Now consider the transformations of the metric tensor:

$$g_{\mu\nu} \longmapsto \overline{g}_{\overline{\mu\nu}} = \frac{\partial x^\mu}{\partial \overline{x}^{\overline{\mu}}} \frac{\partial x^\nu}{\partial \overline{x}^{\overline{\nu}}} g_{\mu\nu}. \tag{2.2}$$

Now, the determinant of the metric and its square root transforms as

$$g \longmapsto \overline{g} = \left| \frac{\partial x^\mu}{\partial \overline{x}^{\overline{\mu}}} \right|^2 g \quad \Longrightarrow \quad \sqrt{-g} \longmapsto \sqrt{-\overline{g}} = \left| \frac{\partial x^\mu}{\partial \overline{x}^{\overline{\mu}}} \right| \sqrt{-g}. \tag{2.3}$$

We know that the four-dimensional volume element transforms as an invariant tensor, i.e. one has the transformation

$$d^4x \longmapsto \left| \frac{\partial \overline{x}^{\overline{\mu}}}{\partial x^\mu} \right| d^4x. \tag{2.4}$$

One can thus assume without any loss of generality, the Jacobian appearing in the transformations to be set to unity, i.e.

$$J := \left| \frac{\partial x^\mu}{\partial \overline{x}^{\overline{\mu}}} \right| \equiv 1. \tag{2.5}$$

This statement is indeed equivalent in setting $\sqrt{-g} = 1$, as the volume element does not have any Jacobians, and thus one can define a perfectly valid volume-preserving transformation. Note that one first has to get the full set of covariant equations which are diffeomorphism invariant, and then restrict or make an appropriate choice of coordinates in which the laws simplify in form.

This approach although simplifies many results, only later it has been realized that assuming such a gauge fixing prior to variation of the Einstein-Hilbert action yields a modification of general relativity. For such realizations, please refer to [10, 11, 12, 21]. Consider the full Einstein-Hilbert action with matter term:

$$S_{\text{EH}} = \frac{1}{2}\int_{\mathcal{M}} d^4x\, \sqrt{-g}\Big(R - 2\Lambda\Big) + S_{\text{m}}, \tag{2.6}$$

where the matter action S_{m} depends on the matter fields, their derivatives, and the metric; and we are using units in which $8\pi G = 1$ unless told otherwise. Now imposing a restricted variation of the metric tensor that preserves the determinant of the metric, i.e. those for which

$$\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g} = 0. \tag{2.7}$$

One can then define an infinitesimal coordinate transformation $x \to x + \xi$, for which the metric transforms as $\delta_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$. Using the restricted variation, one has

$$\delta_\xi\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta_\xi g_{\mu\nu} \equiv 0 \quad \iff \quad \nabla_\mu\xi^\mu = 0. \tag{2.8}$$

These transformations are often referred to as transverse diffeomorphisms or volume-preserving diffeomorphisms. As mentioned above, the equations for general relativity, i.e. Einstein's field equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$ are modified and replaced with the trace-free equations:

$$G_{\mu\nu} - \frac{1}{4}g_{\mu\nu}G = T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T, \tag{2.9}$$

where $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor, and $T_{\mu\nu}$ is the stress-energy-momentum tensor defined as

$$T_{\mu\nu} := \frac{-2}{\sqrt{-g}}\frac{\delta S_{\text{m}}}{\delta g^{\mu\nu}}. \tag{2.10}$$

One wishes to know ways to derive these equations. The most obvious way is to consider adding the unimodular constraint equation $\sqrt{-g} = 1$ into the Einstein-Hilbert action via a Lagrange multiplier $\Lambda(x)$:

$$S_{\text{U}}[g_{\mu\nu}, \Lambda] = \frac{1}{2}\int_{\mathcal{M}} d^4x\, \sqrt{-g}\Big(R - 2\Lambda\Big) + S_{\text{m}} + \int_{\mathcal{M}} d^4x\, \Lambda. \tag{2.11}$$

Notice that in this new action, the cosmological constant is now introduced as a spacetime variable, i.e. as a Lagrange multiplier. Let us rewrite the action in a much more user-friendly fashion to explicitly show the Lagrange multiplier aspect of the cosmological constant:

$$S_{\text{U}} = \int_{\mathcal{M}} d^4x\, \left(\frac{1}{2}\sqrt{-g}R - \Lambda\big(\sqrt{-g} - 1\big)\right) + S_{\text{m}}. \tag{2.12}$$

Variation with respect to Λ immediately leads to the unimodular constraint equation. Variations with respect to the metric tensor lead to Einstein's field equations, but with Λ not constant. Taking the trace of these new field equations with variable cosmological constant gives a value for Λ in terms of the Ricci scalar and the trace of the stress-energy-momentum tensor:

$$-R + 4\Lambda(x) = T \quad \iff \quad \Lambda(x) = \frac{1}{4}\Big(R + T\Big). \tag{2.13}$$

Substituting this back into the modified Einstein's field equations gives the trace-free Einstein's equations given in Eq.(2.9). A rather interesting aspect of these equations is tied to conservation laws. Consider the divergence of the variable Einstein's field equations

$$\nabla^\mu\Big(G_{\mu\nu} + \Lambda(x)g_{\mu\nu}\Big) = \nabla^\mu T_{\mu\nu}. \tag{2.14}$$

Assuming that the stress-energy-momentum tensor is conserved and that the Einstein tensor satisfies the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$, one can safely state the fact that $\nabla^\mu \Lambda \equiv \partial^\mu \Lambda = 0$, assuming that the covariant derivative is a Levi-Civita connection with metricity and torsion-free. The fact that Λ is conserved indicates that it is an integration constant. Thus this modified gravity theory is identically equivalent, at least classically, to ordinary general relativity on-shell. The most important part is the fact that the off-shell action is different from standard general relativity, which is the key point. In this manner, the integration constant suffers from the same radiative instability as the cosmological constant present in general relativity. This version of unimodular gravity does not bring any insight into the cosmological constant problem, nor solves the problem at all, as mentioned in the introduction.

2.1.2 Smolin's and Alternative Approaches

Smolin in [60, 61] proposes an alternative action to define unimodularity. Since the unimodular constraint equation has on its left-hand-side a scalar density and on its right-hand-side an ordinary scalar number being the unity, it does not really make sense to choose such a condition. He proposes to choose a fixed scalar density ϵ_0 for the right-hand-side, namely the unimodular constraint now reads

$$\sqrt{-\overline{g}} = \epsilon_0, \tag{2.15}$$

where $\overline{g}_{\mu\nu}$ is the so-called physical metric given by

$$\overline{g}_{\mu\nu} := \left(\frac{\epsilon_0}{\sqrt{-g}}\right)^{1/2} g_{\mu\nu}. \tag{2.16}$$

With this condition, one can implement the following action instead

$$S_{\text{S}} := \frac{1}{2}\int_{\mathcal{M}} d^4x \; \epsilon_0 \Big(\overline{g}^{\mu\nu} R_{\mu\nu} + \mathcal{L}_{\text{m}(\psi,\overline{g}_{\mu\nu})}\Big), \tag{2.17}$$

where ψ is labeling the set of all possible matter fields. Variation of this action with the standard non-physical metric tensor $g_{\mu\nu}$ gives the trace-free equations

$$R_{\mu\nu} - \frac{1}{4}\overline{g}_{\mu\nu} R = E_{\mu\nu} - \frac{1}{4}\overline{g}_{\mu\nu} E, \tag{2.18}$$

where the source term is given by

$$E_{\mu\nu} := -2\frac{\delta \mathcal{L}_{\text{m}}}{\delta \overline{g}^{\mu\nu}}, \tag{2.19}$$

which differs from the stress-energy-momentum tensor. However, it can be shown that the expression for the energy-momentum tensor is related to this source tensor, namely

$$T_{\mu\nu} = E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{L}_{\text{m}}. \tag{2.20}$$

Thus, upon taking the trace one gets

$$T = E - 2\mathcal{L}_{\text{m}}, \tag{2.21}$$

and thus, one has the following trace-free relations between the source-term and energy-momentum tensor

$$E_{\mu\nu} - \frac{1}{4} g_{\mu\nu} E = T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} T. \tag{2.22}$$

Now taking the divergence of the trace-free equations yields the previously obtained results, namely the fact that the cosmological constant appears as an integration constant.

A crucial difference in comparison to standard general relativity is that any value for Λ is acceptable. Any solution of Einstein's field equations with arbitrary cosmological constant is a solution of the trace-free equations. This fact is due to the invariance of the trace-free equations under the shifts

$$T_{\mu\nu} \longrightarrow T_{\mu\nu} + C g_{\mu\nu}, \tag{2.23}$$

where C is an arbitrary constant. A crucial point is that this invariance is broken once a value for Λ is specified.

Now, consider an alternative action that decouples the quantum vacuum energy contribution. This is given by

$$S = \int_{\mathcal{M}} d^4x \left(\frac{1}{2}\hat{R} - \Lambda\big(\sqrt{-g} - 1\big) \right) + S_{\mathrm{m}}[\psi, \hat{g}_{\mu\nu}], \tag{2.24}$$

with the new physical metric given by

$$\hat{g}_{\mu\nu} := (-g)^{-1/4} g_{\mu\nu}. \tag{2.25}$$

Varying with the unphysical metric leads to the field equations given by

$$\hat{G}_{\mu\nu} - \frac{1}{4}\hat{g}_{\mu\nu}\hat{G} + \Lambda g_{\mu\nu} = \hat{T}_{\mu\nu} - \frac{1}{4}\hat{g}_{\mu\nu}\hat{T}. \tag{2.26}$$

Now using the unimodular constraint in Eq.(2.25), one can remove all hats in the trace-free equations since $\hat{g}_{\mu\nu} = g_{\mu\nu}$. With this taking the trace of the un-hatted trace-free equations gives $\Lambda = 0$. This creates an inconsistency and thus we cannot interpret Λ as a cosmological constant anymore [33].

In the next section, we will review an alternative approach to unimodular gravity which replaces the constant scalar density ϵ_0 or 1 that appears on the right-hand-side of the unimodular constraint with a total four-dimensional total derivative term.

2.2 Henneaux-Teitelboim Gravity

An alternative formulation of unimodular gravity is provided by Henneaux and Teitelboim in [27]. This is a generally covariant theory where the full diffeomorphism group is preserved. The authors first derived it using the Hamiltonian prescription, whereas the Lagrangian perspective is also equivalent, which is what we will be employing throughout this section to review HT-gravity.

2.2.1 Action and Underlying Symmetries

We will assume that there is no matter of any sort for the time being, hence no matter action is assumed to exist: we are in empty vacuum spacetime. The theory is given by three variables, namely the metric tensor $g_{\mu\nu}(x)$, $\Lambda(x)$, and the auxiliary non-dynamical vector density field $\mathcal{T}^\mu(x)$. The action reads

$$S[g_{\mu\nu}, \Lambda, \mathcal{T}^\mu] := \frac{1}{2}\int_{\mathcal{M}} d^4x \left(\sqrt{-g}R - 2\Lambda\left(\sqrt{-g} - \partial_\mu \mathcal{T}^\mu\right) \right). \tag{2.27}$$

Notice that the variable cosmological constant enters the action as a Lagrange multiplier. Variations with respect to it give the unimodular condition

$$\frac{\delta S}{\delta \Lambda} = 0 \quad \Longleftrightarrow \quad \sqrt{-g} = \partial_\mu \mathcal{T}^\mu. \tag{2.28}$$

Now, varying the metric gives the vacuum Einstein's field equation with a spacetime-dependent cosmological constant

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Longleftrightarrow \quad R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\Big(R - 2\Lambda(x)\Big). \tag{2.29}$$

This can be brought back to its standard form by first integrating by parts the action and then varying the vector density, resulting in the on-shell constancy of the cosmological constant parameter

$$\frac{\delta S}{\delta \mathcal{T}^\mu} = 0 \quad \Longleftrightarrow \quad \partial_\mu \Lambda = 0. \tag{2.30}$$

A crucially important aspect of this action is that it has very nice symmetries, one of them being the invariance of the action under the full diffeomorphism group of spacetime transformations. This is unlike other previously mentioned unimodular gravity theories which are only invariant under transverse diffeomorphisms. This means that under an infinitesimal transformation of the spacetime coordinates $x^\mu \mapsto x^\mu + \xi^\mu$, the metric tensor transforms by the Lie derivative of the metric tensor along the vector field ξ^μ:

$$\delta g_{\mu\nu} = (\mathcal{L}_\epsilon g)_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}. \tag{2.31}$$

Λ then transforms as

$$\delta\Lambda = \xi^\mu \partial_\mu \Lambda, \tag{2.32}$$

and $\mathcal{T}^\mu$ transforms according to

$$\delta \mathcal{T}^\mu = \mathcal{T}^\mu \partial_\nu \xi^\nu + 2\xi^{(\nu}\partial_\nu \mathcal{T}^{\mu)}. \tag{2.33}$$

There are other symmetries as well, namely the gauge symmetry $\mathcal{T}^\mu \mapsto \mathcal{T}^\mu + \epsilon^\mu$, where ϵ^μ is a transverse vector field, i.e. $\partial_\mu \epsilon^\mu = 0$. This means that after gauge transformation, the non-dynamical vector field density has one degree of freedom left. This will turn out to be a rather crucial scalar density that when integrated over space gives the so-called unimodular time. Under such a gauge transformation and the fact that $\delta g_{\mu\nu} = \delta\Lambda = 0$ holds, the action remains invariant. The main motivation for why we call this transformation a gauge transformation comes from the fact that one can rewrite the vector density in terms of a three-form field. This can be simply expressed as

$$T_{\alpha\beta\sigma} = \frac{1}{3!}\mathcal{T}^\mu e_{\mu\alpha\beta\sigma}. \tag{2.34}$$

Now, under a $U(1)$-gauge transformation $\chi^{(2)}_{\beta\sigma}$, one has

$$T_{\alpha\beta\sigma} \longmapsto T_{\alpha\beta\sigma} + 3\partial_{[\alpha}\chi^{(2)}_{\beta\sigma]}. \tag{2.35}$$

But this is still reducible, as one can perform the following successive transformations

$$\begin{aligned} \chi^{(2)}_{\alpha\beta} &\longmapsto \chi^{(2)}_{\alpha\beta} + 2\partial_{[\alpha}\chi^{(1)}_{\beta]} \\ \chi^{(1)}_{\alpha} &\longmapsto \chi^{(1)}_{\alpha} + \partial_\alpha \chi^{(0)}. \end{aligned} \tag{2.36}$$

These successive $U(1)$-gauge transformations remove the available free gauge components from four to one, just as the gauge transformed vector density $\mathcal{T}^\mu$. This is the reason why we might expect the action to be invariant under a general $U(1)$-gauge transformation, just as shown above.

Notice that we can re-write the HT action as

$$S_{\text{HT}} = S_{\text{EH}}[g_{\mu\nu}, \Lambda] + \int_{\mathcal{M}} d^4x \; \Lambda \partial_\mu \mathcal{T}^\mu, \tag{2.37}$$

for which upon integrating by parts the second integral term on the left-hand side gives up to a boundary term

$$S_{\text{HT}} = S_{\text{EH}}[g_{\mu\nu}, \Lambda] - \int_{\mathcal{M}} d^4x \; \mathcal{T}^\mu \partial_\mu \Lambda + \text{b.t.} \tag{2.38}$$

Clearly in terms of phase space variables, Λ and $\mathcal{T}^0$ are canonically conjugate pairs. This is again very important as it will lead to the notion of unimodular time mentioned above, which we shall explore more in-depth now.

2.2.2 Unimodular Time and Spacetime Volume

We define the unimodular time as the flux integral

$$T_\Lambda(t) := \int_{\Sigma(t)} d^3x\, \mathcal{T}^0, \tag{2.39}$$

where $\Sigma(t)$ is a spacial Cauchy hypersurface at some coordinate time t. Clearly, the flux that gets in at a given initial-time hypersurface and the flux that gets out at a later-time hypersurface gives a physical measure of spacetime volume. Namely, consider integrating the on-shell unimodular constraint from some initial to final coordinate time defined by the submanifold $\mathcal{I} \subset \mathcal{M}$:

$$\begin{aligned} V \equiv \int_{\mathcal{I}\subset\mathcal{M}} d^4x\, \sqrt{-g} &= \int_{\partial\mathcal{M}} d\Sigma_\alpha \mathcal{T}^\alpha \\ &= \int_{\Sigma(t_f)} d\Sigma_\alpha \mathcal{T}^\alpha - \int_{\Sigma(t_i)} d\Sigma_\alpha \mathcal{T}^\alpha \\ &= \int_{\Sigma(t_f)} d^3x\, \mathcal{T}^0 - \int_{\Sigma(t_i)} d^3x\, \mathcal{T}^0 \\ &= T(t_f) - T(t_i) \equiv \Delta T_\Lambda. \end{aligned} \tag{2.40}$$

Alternatively, one can start by taking the only available gauge freedom in $\mathcal{T}^0$ and applying it to the on-shell unimodular constraint equation

$$\sqrt{-g} = \dot{\mathcal{T}}^0. \tag{2.41}$$

Integrating over the hypersurface at time t gives

$$\int_{\Sigma(t)} \sqrt{-g} = \int_{\Sigma(t)} d^3x\, \dot{\mathcal{T}}^0. \tag{2.42}$$

Further integration between the initial and final times for each hypersurface results in

$$V(t_i \to t_f) \equiv \int_{t_i}^{t_f} dt \int_{\Sigma(t)} d^3x\, \sqrt{-g} = \int_{t_i}^{t_f} dt \int_{\Sigma(t)} d^3x\, \dot{\mathcal{T}}^0 = \Delta T_\Lambda. \tag{2.43}$$

One can thus see that the change in unimodular time is defined as spacetime volume. The converse statement is also true, namely that the spacetime volume created between the two hypersurfaces is due to the change in unimodular time. What this unimodular time actually physically is and why it creates spacetime volume is a rather important question to be answered in the next chapter.

2.2.3 Hamiltonian Analysis

In the previous subsection, we have seen that the vector density field $\mathcal{T}^\mu$ is non-dynamical and does not propagate on-shell, i.e. has a topological nature. This is the most apparent when considering the Hamiltonian formalism to the HT-gravity. This is following the work that appeared in [32].

Consider only the unimodular part of the action split in 3+1:

$$S_{\text{U}}[\Lambda, \mathcal{T}^\mu] = \int dt \int d^3x\, \left(\Lambda\dot{\mathcal{T}}^0 + \Lambda(\partial_i\mathcal{T}^i)\right). \tag{2.44}$$

One can immediately observe that upon integration by parts, $\mathcal{T}^i$ acts as a Lagrange multiplier

$$S_{\text{U}} = \int dt \int d^3x\, \left(\Lambda\dot{\mathcal{T}}^0 - \mathcal{T}^i\partial_i\Lambda\right) + \text{b.t.} \tag{2.45}$$

We can then define the canonical Hamiltonian density

$$\mathcal{H}_{\text{C}} := \mathcal{T}^i \partial_i \Lambda, \tag{2.46}$$

and thus the action takes the rather simple form up to a boundary term

$$S_{\text{U}} = \int dt \int d^3x \left(\Lambda \dot{\mathcal{T}}^0 - \mathcal{H}_{\text{C}}\right). \tag{2.47}$$

It seems rather obvious that Λ and $\mathcal{T}^0$ are canonically conjugate pairs. This can be seen by considering the canonical momenta conjugate to $\mathcal{T}^\mu$:

$$\pi_\mu := \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{T}}^\mu} = \Lambda \delta^0_\mu. \tag{2.48}$$

Following Dirac's prescription on constraint systems, we have four primary constraints

$$\begin{aligned} \phi_0 &:= \pi_0 - \Lambda \approx 0, \\ \phi_i &:= \pi_i \approx 0. \end{aligned} \tag{2.49}$$

The unimodular term can thus be written as

$$S_{\text{U}} = \int dt \int d^3x \left(\pi_0 \dot{\mathcal{T}}^0 - \mathcal{H}_{\text{C}}\right). \tag{2.50}$$

Clearly, variations of the action with respect to the Lagrange multiplier lead to the on-shell spacial constancy of the cosmological constant. Thus the canonical Hamiltonian density is only vanishing on-shell, a very important and crucial point to note. It is thus not a true constraint, and thus we need to find other constraints by extending the phase space. Also, notice that there are no locally propagating degrees of freedom at this point. For this case, the counting goes as

$$\text{d.o.f.} = \frac{2 - 2 \times 1}{2} = 0. \tag{2.51}$$

This can be seen by introducing the constraint $\phi_0 \approx 0$ in the action via a Lagrange multiplier λ^0:

$$S_{\text{U}}^{'} = S_{\text{U}} - \int dt \int d^3x \; \lambda^0 \phi_0, \tag{2.52}$$

with the new Hamiltonian density given as

$$\mathcal{H}^{'} := \mathcal{H}_{\text{C}} + \lambda^0 \phi_0. \tag{2.53}$$

The algebra of constraints satisfied by the constraints are given by the following commuting Poisson brackets:

$$\{\phi_0, \phi_0\} = 0 \qquad \dot{\phi}_0 = \{\phi_0, \mathcal{H}^{'}\}. \tag{2.54}$$

This indicates that $\phi_0 \approx 0$ is a first-class constraint, as required for having no propagating local degrees of freedom.

We can now consider extending the phase space from two dimensions to eight by adding the three other constraints $\phi_i \approx 0$ in the action:

$$\begin{aligned} S_{\text{U}}^{''} &:= \int dt \int d^3x \left(\pi_0 \dot{\mathcal{T}}^0 + \pi_i \dot{\mathcal{T}}^i - \mathcal{H}^{'} - \lambda^i \phi_i\right) \\ &= S_{\text{U}}^{'} + \int dt \int d^3x \left(\pi_i \dot{\mathcal{T}}^i - \lambda^i \phi_i\right) \\ &= \int dt \int d^3x \left(\pi_0 \dot{\mathcal{T}}^0 + \pi_i \dot{\mathcal{T}}^i - \mathcal{H}^{''}\right), \end{aligned} \tag{2.55}$$

where now the new Hamiltonian density reads

$$\mathcal{H}^{''} := \mathcal{H}^{'} + \lambda^i \phi_i. \tag{2.56}$$

In this case, there are four first-class constraints and no second-class constraints. This is indicated by the algebra

$$\begin{aligned}
&\{\phi_0, \phi_0\} = 0 \qquad \{\phi_i, \phi_j\} = 0, \\
&\{\phi_0, \phi_i\} = 0, \\
&\quad \dot{\phi}_0 = \{\phi_0, \mathcal{H}^{''}\} = 0, \\
&\quad \dot{\phi}_i = \{\phi_i, \mathcal{H}^{''}\} = -\partial_i \Lambda \equiv 0.
\end{aligned} \tag{2.57}$$

Thus, the counting of local degrees of freedom goes as

$$\text{d.o.f.} = \frac{8 - 2 \times 4 - 0}{2} = 0. \tag{2.58}$$

We thus have again no local propagation.

Finally, since $\mathcal{T}^i$ appears as a Lagrange multiplier only, we can gauge fix it and thus add it as a constraint $\mathcal{T}^i \approx 0$ in the action. The final form of the extended action thus takes the rather interesting form

$$S_{\text{U}}^{'''} := S_{\text{U}}^{''} - \int dt \int d^3x \; \mu_i \mathcal{T}^i, \tag{2.59}$$

where μ_i is a Lagrange multiplier. The rather important fact that this constraint is of second class indicates that the counting of local propagating degrees of freedom will now have second-class constraints as well. We thus have

$$\text{d.o.f.} = \frac{8 - 2 \times 1 - 6}{2} = 0. \tag{2.60}$$

The Poisson bracket algebra goes as

$$\begin{aligned}
&\{\phi_0, \phi_0\} = 0 \quad \{\phi_0, \mathcal{H}^{''}\} = 0 \\
&\{\phi_i, \phi_j\} = 0 \quad \{\phi_i, \mathcal{H}^{''}\} \approx 0 \\
&\{\phi_i, \phi_0\} = 0 \quad \{\mathcal{T}^i, \mathcal{H}^{''}\} \approx 0 \\
&\{\phi_i, \mathcal{T}^j\} = \delta_i^j
\end{aligned} \tag{2.61}$$

We thus see that, unlike the previous approaches, this time $\phi_i \approx 0$ constraint gets promoted from first-class to second-class constraint.

Overall, within each method, no new local degrees of freedom emerge, a matter due to the structure of constraints, which points to the underlying gauge symmetry of the theory, as well as the topological nature of the theory, which we shall take seriously and explore its consequences in the next chapter.

2.2.4 Differential Geometric Formulation

For completeness and pure pleasure, let us also reformulate HT-gravity using differential forms, as we believe it is more elegant and simple to see everything in terms of geometrical objects.

Let $\mathcal{M}$ be a globally hyperbolic four-dimensional spacetime manifold equipped with metric g. Global hyperbolicity is assumed here since it is always possible to foliate such spacetime manifolds, provided that there is no torsion present. We are using conventions in which the Levi-Civita tensor and symbol are given as

$$\epsilon_{\alpha\beta\mu\nu} := \sqrt{-g}\; e_{\alpha\beta\mu\nu}, \qquad \epsilon^{\alpha\beta\mu\nu} := \frac{1}{\sqrt{-g}} e^{\alpha\beta\mu\nu}, \tag{2.62}$$

and for which the top form skew-symmetric tensor basis $dx^\alpha \wedge dx^\beta \wedge dx^\mu \wedge dx^\nu$ is

$$dx^\alpha \wedge dx^\beta \wedge dx^\mu \wedge dx^\nu := -e^{\alpha\beta\mu\nu}\ dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \equiv -e^{\alpha\beta\mu\nu}\ d^4x. \tag{2.63}$$

The volume form on $\mathcal{M}$ being a special top form is given by

$$\begin{aligned}\Omega_g &:= \frac{1}{4!}(\Omega_g)_{\alpha\beta\mu\nu}\ dx^\alpha \wedge dx^\beta \wedge dx^\mu \wedge dx^\nu \\ &= -\frac{1}{4!}\sqrt{-g}\ e_{\alpha\beta\mu\nu}e^{\alpha\beta\mu\nu}\ d^4x \\ &= \sqrt{-g}\ d^4x.\end{aligned} \tag{2.64}$$

We also have the non-propagating and non-dynamical three-form field T written as

$$T := \frac{1}{3!}T_{\alpha\beta\sigma}\ dx^\alpha \wedge dx^\beta \wedge dx^\sigma. \tag{2.65}$$

We can thus write the derivative term appearing in the unimodular action given by Eq.(2.37) as

$$\partial_\mu \mathcal{T}^\mu\ d^4x = -dT, \tag{2.66}$$

and thus the unimodular action reads

$$S_{\text{HT}} := \frac{1}{2}\int_{\mathcal{M}} \Omega_g\Big(R - 2\Lambda\Big) - \int_{\mathcal{M}} \Lambda dT. \tag{2.67}$$

This can also be seen by introducing a top form that is also an exact form, namely, i.e. $B = dT$. The action then takes the rather simple form

$$S_{\text{HT}} := \frac{1}{2}\int_{\mathcal{M}} \Omega_g\Big(R - 2\Lambda\Big) - \int_{\mathcal{M}} \Lambda B. \tag{2.68}$$

The action has the same symmetry as a $U(1)$ gauge theory, i.e. the successive chain of transformations

$$\begin{aligned}T &\longmapsto T + d\chi^{(2)}, \\ \chi^{(2)} &\longmapsto \chi^{(2)} + d\chi^{(1)}, \\ \chi^{(1)} &\longmapsto \chi^{(1)} + d\chi^{(0)}.\end{aligned} \tag{2.69}$$

Finally, the unimodular constraint equation can simply be recast as

$$\Omega_g = -dT, \tag{2.70}$$

and the on-shell constancy of the cosmological constant is

$$d\Lambda = 0. \tag{2.71}$$

Obviously, Einstein's equations do not change in form as they are already geometrical.

In the next section, we will expand on the fact that not only the cosmological constant can be taken on-shell constant, but other constants of Nature can also be taken as variables, provided they are conserved on-shell.

2.3 Deconstantizing Natures Constants

So far we have only considered the cosmological constant as a variable constant in the action, and have shown that it can be made arbitrarily constant on-shell, i.e. as an equation of motion. What if we have other constants that we wish to promote to non-fixed variables? This particular question has been explored in a series of papers [47, 48, 49], and we shall review these results in this section.

The first step is to consider a base theory described by an action S_0. This can be the standard Einstein-Hilbert action (with or without Λ) or any other alternative theory of gravity. To this, we add the unimodular term, which to every constant promoted to a field variable $\boldsymbol{\alpha}$, there is an associated vector density field $\mathcal{T}^\mu_{\boldsymbol{\alpha}}$. The action for such a model is

$$S = S_0 + \int d^4x \, \boldsymbol{\alpha} \cdot \partial_\mu \mathcal{T}^\mu_{\boldsymbol{\alpha}}. \tag{2.72}$$

The action is again invariant under the full diffeomorphisms group of spacetime coordinate transformations, and hereby generalized including all constants. These can be used to define canonical phase space pairs $(\boldsymbol{\alpha}, \mathcal{T}^0_{\boldsymbol{\alpha}})$ and each $\boldsymbol{\alpha}$ has an associated on-shell constancy. Variations of the action with respect to these non-fixed parameters lead to

$$\frac{\delta S}{\delta \boldsymbol{\alpha}} = 0 \implies \dot{\mathcal{T}}^0_{\boldsymbol{\alpha}} = \frac{\delta S_0}{\delta \boldsymbol{\alpha}}. \tag{2.73}$$

In the 3+1 splitting, this equation can be integrated over each hypersurface $\Sigma(t)$ with $t_i \leq t \leq t_f$, to give a physical time for each $\boldsymbol{\alpha}$:

$$\begin{aligned} \Delta T_{\boldsymbol{\alpha}} &= T_{\boldsymbol{\alpha}}(t_f) - T_{\boldsymbol{\alpha}}(t_i) \\ &= \int_{t_i}^{t_f} dt \int_{\Sigma(t)} d^3x \, \frac{\delta S_0}{\delta \boldsymbol{\alpha}} \\ &= I_{\boldsymbol{\alpha}}(t_i \to t_f), \end{aligned} \tag{2.74}$$

where $I_{\boldsymbol{\alpha}}(t_i \to t_f)$ is the base action integral without the constants from an initial time to a final time. The times $T_{\boldsymbol{\alpha}}$ are given by

$$T_{\boldsymbol{\alpha}}(t) := \int_{\Sigma(t)} d^3x \, \mathcal{T}^0_{\boldsymbol{\alpha}}. \tag{2.75}$$

To be more precise, the physical times depend on the spacetime foliation hypersurfaces Σ, which themselves depend on coordinate time t. Thus the nature of $T_{\boldsymbol{\alpha}}$ is highly observer-dependent, i.e. the way we define the foliation.

Consider the illustrative case of the base theory as the Einstein-Hilbert action with Λ, where the gravitational constant G is restored. This theory will turn out to be rather interesting in the case of the axion-type topological term. The two-dimensional vector of constants is chosen to be

$$\boldsymbol{\alpha} = \left(\frac{\Lambda}{8\pi G}, \frac{1}{8\pi G} \right) \equiv (\alpha_1, \alpha_2). \tag{2.76}$$

Note that α_1 is the Λ-vacuum energy density ρ_Λ, and α_2 is the reduced Planck mass squared (or Planck energy) M_{P}^2. The extended-unimodular action then takes the rather simple form

$$S = \int d^4x \, \alpha_2 \left(\frac{1}{2}\sqrt{-g}R + \dot{\mathcal{T}}^0_{\alpha_2} \right) + \int d^4x \, \alpha_1 \left(-\sqrt{-g} + \dot{\mathcal{T}}^0_{\alpha_1} \right). \tag{2.77}$$

From Eqs.(2.73-2.74), we obtain the following time differences

$$\begin{aligned} \Delta T_{\alpha_1} &= I_{\alpha_1}(t_i \to t_f), \\ \Delta T_{\alpha_2} &= -I_{\alpha_2}(t_i \to t_f). \end{aligned} \tag{2.78}$$

The first integral I_{α_1} is

$$I_{\alpha_1} := \int d^4x \, \sqrt{-g} \equiv V, \tag{2.79}$$

i.e. the four-volume of spacetime from initial and final times. Then we shall refer to I_{α_2} as the Ricci-volume, since it is a four-volume weighted by the Ricci scalar:

$$I_{\alpha_1} := \frac{1}{2}\int d^4x\ \sqrt{-g}R. \tag{2.80}$$

It is rather interesting that the ratios of Eqs.(2.79-2.80) is the spacetime average of the Ricci scalar between initial and final times

$$\overline{R}(t_i \to t_f) = -\frac{\Delta T_{\alpha_1}}{\Delta T_{\alpha_2}}. \tag{2.81}$$

This gives a Universe with two ages, ΔT_{α_1} and ΔT_{α_2}, the Ricci-volume and four-volume times, respectively. In [38, 47], the observed value of the cosmological constant is identified to be

$$\Lambda_{\text{obs}} = \frac{\overline{R}(t_i \to t_f)}{4} = -\frac{1}{4}\frac{\Delta T_{\alpha_1}}{\Delta T_{\alpha_2}}. \tag{2.82}$$

The value of the observed cosmological constant has to be understood as a measure of ratios of times, and for that one has to understand what these times are.

Consider a set of fixed constants $\boldsymbol{\beta}$ such that each evolves according to various unimodular times $T_{\boldsymbol{\alpha}}$ by demanding

$$\boldsymbol{\beta} = \boldsymbol{\beta}(T_{\boldsymbol{\alpha}}). \tag{2.83}$$

There may be overlaps between each set of constants $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. This amounts to introducing second-class constraints, which we shall not be concerned about in this thesis. Instead, we assume that these constants are separate, with the $\boldsymbol{\alpha}$ deconstantized and conjugate to their associated unimodular times, while the constants $\boldsymbol{\beta}$ will depend on these times as above. This can be the case for instance when the Λ-energy density is deconstantized, while the squared Planck mass is a genuine constant that evolves with the four-volume time. We will not consider such constructions.

Lastly, the canonical phase space variables of unimodular gravity satisfy Heisenberg uncertainty relations:

$$\begin{aligned}\sigma(\Lambda)\ \sigma\left(\int_V d^4x\ \sqrt{-g}\right) &\geq 4\pi l_{\text{P}}^2,\\ \sigma(G)\ \sigma\left(\int_V d^4x\ \sqrt{-g}R\right) &\geq 4\pi G l_{\text{P}}^2.\end{aligned} \tag{2.84}$$

Near spacetime singularities, the presence of these inequalities possibly implies quantum mechanics dominate. Another interesting fact is that these uncertainty relations are in terms of the Planck area l_{P}^2, which exhibits close similarities with canonical quantum gravity theories.

2.4 Extension via Proca Mass Term

We would like to close this chapter by making a short review of the new results found in [32] regarding the fact that adding a mass term to unimodular gravity gives local propagation of the vector density.

For starters, consider the vacuum Maxwell's action in four-dimensional Minkowski spacetime

$$S_{\text{M}} := -\frac{1}{4}\int d^4x\ F_{\mu\nu}F^{\mu\nu}, \tag{2.85}$$

where $F = dA$ is the electromagnetic field strength curvature two-form and A is the electromagnetic gauge field one-form. This action has the very nice $U(1)$-gauge symmetry, namely transformations $A \to A + d\eta^{(0)}$, with $\eta^{(0)}$ a smooth function, leaves the action invariant. There are numerous ways of breaking this symmetry, one of them accomplished by adding a so-called Proca mass term, i.e.

$$S_{\text{P}} := S_{\text{M}} - \int d^4x\ \frac{m^2}{2}A_\mu A^\mu. \tag{2.86}$$

Restoring back the symmetry amounts to performing a Stuckelberg trick.

Now, we can rewrite the HT-gravity action in a fully covariant fashion by introducing the genuine vector field

$$T^\mu := (-g)^{-1/2}\mathcal{T}^\mu. \tag{2.87}$$

With this, the unimodular condition reads simply

$$\nabla_\mu T^\mu = 1. \tag{2.88}$$

The unimodular part of the HT-gravity action then takes the explicitly covariant form

$$S_{\text{U}} = \int d^4x\ \sqrt{-g}\Lambda\nabla_\mu T^\mu. \tag{2.89}$$

Consider breaking the $U(1)$ gauge symmetry of unimodular gravity by considering a similar Proca-type mass term. The new action will read

$$S = S_{\text{U}} + \frac{M^2}{2}\int d^4x\ \sqrt{-g}T_\mu T^\mu. \tag{2.90}$$

The sign difference compared to the standard electromagnetic Proca term is because we wish to give positive energy to the scalar mode. One can also write up to boundary term after integration by parts

$$S = -\int d^4x\ \sqrt{-g}\Big((\partial_\mu\Lambda)T^\mu - \frac{M^2}{2}T_\mu T^\mu\Big) + \text{b.t.} \tag{2.91}$$

Let us evaluate the equations of motion for the unimodular term. Variations with Λ is the same as Eq.(2.88). However, the good news is obtained upon variations with respect to the vector field:

$$\partial_\mu\Lambda = M^2 T_\mu. \tag{2.92}$$

Thus taking a covariant derivative results in a wave equation for Λ:

$$\Box\Lambda = M^2. \tag{2.93}$$

This equation can be easily solved by taking Λ as a plane wave solution. It is also a linear equation, i.e. one can take $\Lambda = \Lambda_0 + \zeta$ such that $\Box\zeta = 0$ and $\Box\Lambda = M^2$. Choosing Λ_0 homogeneous on each hypersurface defines a gauge-invariant zero-mode for the unimodular time. Thus, the extra field ζ is a massless scalar field, a Lambdon to be precise if we were to quantize the theory. This is the new local degree of freedom that has been released by breaking the abelian gauge invariance of HT-gravity. In fact, computing the energy-momentum tensor for this model results in

$$T_{\mu\nu} := \frac{-2}{\sqrt{-g}}\frac{\delta S_{\text{Mass}}}{\delta g^{\mu\nu}} = \frac{1}{M^2}\Big(\partial_\mu\Lambda\partial_\nu\Lambda - \frac{1}{2}g_{\mu\nu}\partial_\alpha\Lambda\partial^\alpha\Lambda\Big). \tag{2.94}$$

As it can be clearly seen, this is nothing but the energy-momentum tensor for a scalar field Λ/M.

The main point to get from this chapter is that there exist various approaches towards unimodularity in gravitational theories and they all possess a topological aspect, namely, the theory does not have local propagating degrees of freedom besides the metric tensor field. These facts will be taken very seriously in the next chapter and will pave the door towards a new framework for relational times within gravity, a long chased problem in quantum gravity theories.

Chapter 3

Topology of Time

The subject of topology is one of the most important (if not the most important) branches of mathematics that has applications in almost anything to do with physics. As was shown in the previous chapter within various approaches to unimodular gravity, all had one common property, that is there were no locally propagating degree of freedom for the vector density, unless some mass term is added, which brakes the inherent gauge symmetry of the action. One has to investigate the implications of such a statement, namely the topological aspect of the theory. The question of how can this be made more explicit is at the origin of how this chapter will turn out and will explain some of the mysteries we asked previously: what is this unimodular time really about? What does it tell us?

To answer this question, we will first have to introduce some hardcore mathematical results achieved over the past few decades. We will start off by reviewing results in Chern-Simons theory purely classically, and then dive into the realm of knot theory. Afterwards, we will focus on how these tie with unimodular gravity by modifying the HT-gravity action to include a Chern-Simons-inspired term (called the Pontryagin term) and show how the unimodular time has a topological side to it.

3.1 Witten and Chern-Simons Theory

In 1989, Edward Witten published a major breakthrough in mathematical physics [68] that made it possible to obtain the Fields Medal award one year later. In this section, we will focus on some aspects of this paper and provide some explanations for how it will be used within unimodular gravity. A note here to be mentioned is to refer to the Appendix for more details about fibre bundle formalism, as it will be used here with the assumption that some basics about fibre bundles are known.

3.1.1 Characteristic Classes

Results follow from [1]. The main idea is to classify the possible bundles that can be built from a given fibre, base space and total space using cohomology. We can thus use homotopy to classify the twisting of fibres and see links between these concepts within characteristic classes.

Let $\omega \in \Omega^k(\mathcal{M}) \otimes \mathfrak{g}$ be a Lie algebra-valued k-form, i.e. one can write $\omega = \eta \otimes \tilde{\omega}$ with $eta \in \Omega^k(\mathcal{M})$ and $\tilde{\omega} \in \mathfrak{g}$. Let P be a polynomial defined as the mapping

$$P : \Omega^k(\mathcal{M}) \otimes \mathfrak{g} \longrightarrow \Omega^{kr}(\mathcal{M}), \tag{3.1}$$

such that $P(\omega) := P(\tilde{\omega}) \underbrace{\eta \wedge ... \wedge \eta}_{\text{r}}$. Here, $P(\tilde{\omega})$ can be regarded as a number over the real or complex fields.

An invariant polynomial is invariant with respect to the adjoint action of a Lie group G if

$$P(\omega) = P(g^{-1}\omega g), \quad \forall g \in G. \tag{3.2}$$

We are interested in polynomials of the curvature two-form $F = dA + A \wedge A$, with A as the gauge connection defined over the principal G-bundle. This is the case as under a gauge transformation, F transforms in the adjoint representation i.e. $F \to g^{-1}Fg$.

We would like to give a theorem relating invariant polynomials to cohomology, the so-called Chern-Weil theorem. Let P be an invariant polynomial. Then $P(F)$ satisfies the following two properties:

- $dP(F) = 0$,
- $P(F) = P(F') = d\chi$, where F and F' are curvatures two-forms of two different connections A and A' on a principal G-bundle.

The proof can be found in Nakahara [53]. This theorem tells us that we can associate a cohomology class to invariant polynomials which does not depend on the choice of a connection on the bundle. We can thus say that topologically equivalent bundles associated with the same cohomology class of the base space $\mathcal{M}$ are represented by invariant polynomials.

A characteristic class of a fibre bundle over $\mathcal{M}$ is the cohomology class of $\mathcal{M}$ corresponding to a given invariant polynomial P. Characteristic classes are topological invariants, i.e. topologically equivalent bundles have the same characteristic classes. We can thus classify fibre bundles according to their characteristic classes! There exist various such characteristic classes. We will focus on the Chern and Pontryagin classes.

Chern Classes

Let $(E, \mathcal{M}, \pi)$ be a complex vector bundle with fibres $F \cong \mathbb{C}^k$ with structure group $G \subset GL(k, \mathbb{C})$. We define the total Chern class

$$c(F) := \det\left(\mathbb{I} + i\frac{F}{2\pi}\right). \tag{3.3}$$

This can be expanded as a sum of even-degree forms

$$c(F) = 1 + c_1(F) + c_2(F) + ..., \tag{3.4}$$

where $c_i(F) \in H^{2i}(\mathcal{M})$ is the ith Chern-class given explicitly by

$$\begin{aligned} c_0(F) &= 1, \\ c_1(F) &= \frac{i}{2\pi}\mathrm{Tr}[F], \\ c_2(F) &= \frac{1}{2}\left(\frac{i}{2\pi}\right)^2\Big(\mathrm{Tr}[F]\wedge\mathrm{Tr}[F] - \mathrm{Tr}[F\wedge F]\Big), \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ c_k(F) &= \left(\frac{i}{2\pi}\right)^k \det(F). \end{aligned} \tag{3.5}$$

Pontryagin Classes

Now consider $(E, \mathcal{M}, \pi)$ a real vector bundle with fibres $F \cong \mathbb{R}^k$ with structure group $G \subset GL(k, \mathbb{R})$. With metrisable fibres F, one can further reduce the gauge group to the orthogonal group $O(k)$. The total Pontryagin class is then defined by

$$p(F) := \det\left(\mathbb{I} + \frac{F}{2\pi}\right). \tag{3.6}$$

Since the gauge group is the orthogonal group, we have antisymmetry and thus we can expand the polynomial in even functions of the curvature two-form, namely

$$p(F) = 1 + p_1(F) + p_2(F) + ... \tag{3.7}$$

Thus $p_i(F)$ is a polynomial of order-$2i$.

This closes the lightning review on characteristic classes. They turn out to be very useful when constructing topological field theories as we shall see in the next subsections.

3.1.2 θ-term in $d = 4$ Yang Mills

Consider a Yang-Mills theory in four-dimensional Euclidean spacetime with gauge group $SU(2)$. The action reads

$$S = S_{\text{YM}} + S_\theta, \tag{3.8}$$

where the Yang-Mills term in the action is

$$S_{\text{YM}} := \frac{1}{g^2} \int \text{Tr}[F \wedge *F], \tag{3.9}$$

and the θ-term in the action is

$$S_\theta := \frac{\theta}{8\pi^2} \int \text{Tr}[F \wedge F]. \tag{3.10}$$

Note that for $U(1)$ gauge group, one needs the gauge theory to have normalisation $\theta/4\pi^2$, as this is necessary for which $\theta \sim \theta + 2\pi$ in abelian and non-abelian cases. For the gauge group $SU(n)$, one has Tr$[F] = 0$, i.e.

$$S_\theta = \theta \int c_2(F). \tag{3.11}$$

Notice that in the path integral, e^{iS_θ} is invariant under 2π-shifts in the θ-parameter.

Interestingly enough, taking the differential of Tr$[F \wedge F]$ is zero, i.e. we can define a three-form K called the Chern-Simons form given by

$$K := \text{Tr}\left[A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right] \quad \Longleftrightarrow \quad dK = \text{Tr}[F \wedge F]. \tag{3.12}$$

We can thus rewrite the θ-term entirely in terms of this Chern-Simons three-form, integrated over the three-sphere at the boundary of the spacetime manifold, that is

$$\begin{aligned} S_\theta &= \frac{\theta}{8\pi^2} \int_{\mathbb{E}^4} \text{Tr}[F \wedge F] \\ &= \frac{\theta}{8\pi^2} \int_{S^3_\infty} \text{Tr}\left[A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right]. \end{aligned} \tag{3.13}$$

This term is non-zero if the connection is to be non-trivial at the boundary. This implies that finite energy solutions have the curvature tending to zero at infinity: this is the case of pure gauge

$$A \longrightarrow g^{-1} dg \;\; \text{as} \;\; x \longrightarrow \infty, \tag{3.14}$$

where $g : S^3 \to SU(2)$, which are classified by homotopy groups that we shall come shortly. Thus finite energy solutions have the θ-term action given by

$$S_\theta = \frac{\theta}{24\pi^2} \int_{S^3_\infty} \text{Tr}[(g^{-1} dg)^3]. \tag{3.15}$$

The integral term without the θ-parameter is the so-called winding number w and it is an integer as we shall see now.

Homotopy and Winding Number

An important aside is as promised above on homotopy and its relation to winding number. Let X be any topological space and $\pi_n(X)$ be the set of equivalence classes of maps from $S^n \to X$ that cannot be smoothly deformed into each other. This set is called the nth-homotopy group of X. There exists a fundamental result in the homotopy theory that states that for all simple compact Lie groups G, $\pi_3(G) \cong \mathbb{Z}$. The canonical case is for instance when the Lie group is $G = SU(2)$ which is isomorphic to the three-sphere S^3:

$$\pi_3(S^3) \cong \mathbb{Z}. \tag{3.16}$$

In fact, there exists a general result regarding the nth homotopy group of the n-sphere:

$$\pi_n(S^n) \cong \mathbb{Z}, \quad \forall n \text{ integer}. \tag{3.17}$$

Thanks to the representation theory of Lie groups and Lie algebras, for other gauge groups G, it is sufficient to just consider $SU(2)$ subgroups: winding in $SU(2)$ stays and all other windings of other groups G can be deformed to be the winding of $SU(2)$.

The question of what counts the winding in gauge groups is a rather important aspect to bring clarity. Let X, Y be two oriented closed manifolds of the same dimension, and let $\Omega \in \Omega^{\text{dim}Y}(Y)$ be the volume form on Y normalised such that $\int_Y \Omega = 1$. Note that if Y is a Lie group, then one can use the so-called Haar measure as the integration. Let also the function f be defined as the mapping $f : X \to Y$. Then, we define the topological degree of the function f as

$$\text{deg}(f) := \int_X f^*(\Omega) \in \mathbb{Z}. \tag{3.18}$$

A nice feature of topological degree mapping is that it is a topological invariant. It also does not depend on the choice of the volume form Ω. For instance, take $Y = SU(2)$. Then the standard Haar measure read

$$\Omega = \frac{1}{24\pi^2}\text{Tr}[(h^{-1}dh)^3], \quad h \in SU(2). \tag{3.19}$$

This implies that $int_S^3 \Omega = 1$. Now, if $X = SU(2)$, then with the function $f : S^3 \to S^3$, we have

$$\begin{aligned} \text{deg}(f) &= \int_{S^3} f^*(\Omega) \\ &= \frac{1}{24\pi^3} \int_{S^3} \text{Tr}[(g^{-1}dg)^3] \equiv w, \end{aligned} \tag{3.20}$$

where $g : S^3 \to S^3$. We thus have an integer for the winding number from the way the topological degree is defined. Note that since $pi_3(G) \cong \mathbb{Z}$, the same applies for larger G and we need to consider $SU(2)$. For example, with $G = SU(2)$ the map

$$g(x) = \mathrm{e}^{\frac{i}{2}\omega(r)\sigma_i \hat{x}^i}, \tag{3.21}$$

with $\omega(r)$ monotonic such that $\omega = 0$ for $r = 0$ and $\omega = 4\pi n$ for $r = \infty$, has topological degree $w = n$.

3.1.3 Yang-Mills Instantons

We have seen in the previous subsection that there exist topologically distinct solutions. We can also construct vacuum solutions, namely find solutions to the equations of the sort $D * F = 0$, with a non-trivial winding number. Such solutions are called instantons. The action takes the form

$$\begin{aligned} S_{\text{YM}} &= \frac{1}{g^2}\int \text{Tr}[F \wedge *F] \\ &= \underbrace{\frac{1}{2g^2}\int \text{Tr}[(F - *F)^2]}_{\geq 0} + \frac{1}{g^2}\underbrace{\int \text{Tr}[F \wedge F]}_{=8\pi^2 w} \geq \frac{8\pi^2 w}{g^2}. \end{aligned} \tag{3.22}$$

This inequality is known as the Bogomolny energy bound. We can see that this action is bounded and minimised when $F = *F$. These equations are the self-duality conditions and are first-order differential equations. Notice that with the self-duality condition, we have the Bianchi identity automatically satisfied

$$D \wedge F = DF = 0. \tag{3.23}$$

3.1.4 Chern-Simons Action

We are now fully ready to construct the Chern-Simons action. The main mathematical results were obtained in [13, 14].

Let $P_i \in H^{2i}(\mathcal{M}, \mathbb{Z})$ be a characteristic class, i.e. one has the conditions that $P_i(F)$ is a closed form by the Chern-Weyl theorem, and that locally one can always write

$$P_i(F) = dK_{2i-1}(F, A). \tag{3.24}$$

Now suppose that we have $2i$-dimensional manifold $\mathcal{M}$ with boundary. Then by Stokes theorem, one can write

$$\int_{\mathcal{M}} P_i(F) = \int_{\partial\mathcal{M}} K_{2i-1}(F, A). \tag{3.25}$$

Now, the left-hand side is an integer, which means that the Chern-Simons form is also a characteristic class, i.e. also describes the topology of the boundary. For Chern classes, one finds that

$$\begin{aligned}
K_1(F, A) &= \frac{i}{2\pi}\mathrm{Tr}[A] \\
K_3(F, A) &= \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \mathrm{Tr}\left[A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right] \\
K_5(F, A) &= \frac{1}{6}\left(\frac{i}{2\pi}\right)^3 \mathrm{Tr}\left[A \wedge dA \wedge dA + \frac{3}{2}A \wedge A \wedge A \wedge dA + \frac{3}{5}A \wedge A \wedge A \wedge A \wedge A\right] \\
&\;\;. \\
&\;\;. \\
&\;\;.
\end{aligned} \tag{3.26}$$

As one can see, the Chern-Simons forms have odd degrees, which means that they are usually defined in odd dimensions. Another important remark is that Chern-Simons theories are topological: the stress-energy tensor vanishes and we do not have a length scale, that is, Wilson loops do not have a so-called area law.

Non-Abelian Chern-Simons in $d = 3$

Consider a three-dimensional Chern-Simons theory on the three-sphere, with level k. The action reads

$$S := \frac{k}{4\pi}\int_{S^3} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right). \tag{3.27}$$

Now, under a gauge transformation $A \mapsto g^{-1}Ag + g^{-1}dg$, where $g(x) \in SU(2)$, the action above transforms as

$$S \longmapsto S + \underbrace{\frac{k}{4\pi}\int_{\partial S^3} \mathrm{Tr}[g^{-1}dg \wedge A]}_{\to 0} + \frac{k}{12\pi}\int_{S^3} \mathrm{Tr}[g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg]. \tag{3.28}$$

The second integral goes to zero as the three-sphere does not have a boundary. Now recall that the set of transformations g are characterised by the 3rd-homotopy group of the three-sphere, which are the integers, are labelled by the winding number, as defined in Eq.(3.20). Thus, under a gauge transformation of the Chern-Simons action, we get

$$S \longmapsto S + 2\pi kw. \tag{3.29}$$

Note that again in the path integral, we require that $k \in \mathbb{Z}$, otherwise the exponential e^{iS} is not well-defined.

Abelian Chern-Simons theory in $d = 3$

For the abelian case, we can consider the gauge group $U(1)$ with level k. The Chern-Simons action in three dimensions takes the rather simple form

$$S = \frac{k}{4\pi} \int_{S^3} A \wedge dA \equiv \frac{k}{4\pi} \int_{S^3} A \wedge F, \tag{3.30}$$

where for a $U(1)$ gauge connection A, the curvature two-form reads $F = dA$ for a contractible Riemannian manifold. In fact, one can split the three-sphere and rewrite it as a product manifold $S^1 \times S^2$, where physically the time manifold has been compactified to a circle and the space manifold is identified with the two-sphere.

Now consider a gauge transformation $A \mapsto A + d\chi^{(0)}$, the action transforms as

$$\begin{aligned} S \longmapsto & \frac{k}{4\pi} \int_{S^3} (A + d\chi^{(0)}) \wedge d(A + d\chi^{(0)}) \\ &= \frac{k}{4\pi} \int_{S^3} (A + d\chi^{(0)}) \wedge dA \\ &= \frac{k}{4\pi} \int_{S^3} A \wedge dA + \frac{k}{4\pi} \int_{S^3} d\chi^{(0)} \wedge dA \\ &= S + \underbrace{\frac{k}{4\pi} \int_{S^3} d(\chi^{(0)} dA)}_{\to 0} \\ &= S. \end{aligned} \tag{3.31}$$

We thus see that, unlike the non-abelian case, the action is invariant under the gauge transformation for the case when the 3-manifold has no boundary. A key result to note is that we considered little gauge transformations, and only those leave the action invariant. If we consider large gauge transformations of the type $A \mapsto A + \omega$ where ω is a closed one-form that is not necessarily globally exact, then the action does not remain invariant, but will pick up an extra term, which is not the winding number but

$$S \longmapsto S + 2\pi k N, \tag{3.32}$$

where $k \in \mathbb{Z}$ and $N \in \mathbb{Z}$. This comes from the definition of the flux integral on the two-sphere:

$$\int_{S^2} F = 2\pi N. \tag{3.33}$$

Generalities on Wilson Lines and Loop Operators

According to quantum field theory rules, one could study the partition function for any given Chern-Simons theory. Yet, we wish to construct a suitable class of observables that are metric-independent and gauge-invariant. Local gauge invariant operators would not be appropriate as they spoil general covariance [68]. It turns out that one can define appropriate observables on any three-manifold. These appeared in quantum chromodynamics and are called Wilson lines. The results follow closely [68, 9].

Let C be an oriented closed curve in $\mathcal{M}$. We should point out that such a curve C is an embedding of a circle S^1 into the manifold $\mathcal{M}$: this is the first time we will use the terminology knot. Consider also R to be an irreducible representation of the gauge group G. One can then define the Wilson line $W_R(C)$ as the functional of the gauge connection A:

$$W_R(C) := \mathrm{Tr}_R \Big[\mathcal{P} \, \mathrm{e}^{\int_C A} \Big]. \tag{3.34}$$

We have several important concepts going into this expression. We first compute the holonomy of the gauge connection A around the knot C, getting an element of the gauge group G which is well-defined up to conjugacy, and then one takes the trace of this element in the irreducible representation R of the gauge group. As can be seen from the above expression, there is no need to introduce a metric, and thus general covariance is maintained. For the case where the gauge group is of a non-abelian nature, we evaluate a path ordering $\mathcal{P}$ inside the trace.

With the Wilson lines defined, we can formulate the general problem of interest, that is we consider an oriented 3-manifold $\mathcal{M}$, take r-oriented and non-intersecting knots $C_i,\ i = 1, ..., r$, whose union is what is known in the literature of knot theorists as a link L. Assigning a representation R_i to each knot C_i, we can then evaluate the integral observable

$$Z(\mathcal{M}; C_i, R_i) := \int DA\ \mathrm{e}^{iS} \prod_{i=1}^{r} W_{R_i}(C_i), \tag{3.35}$$

where the integration measure represents Feynman's integral over all gauge orbits, that is an integral over all equivalence classes of connections modulo gauge transformations.

3.1.5 Knot Polynomials from Chern-Simons Theory

We are now ready to explain what Witten did with Chern-Simons theory and how it can be used to compute various knot polynomials that appear in mathematical contexts, such as in knot theory. We are not going too deep into the computations but only getting the idea right so that we can later apply the ideas within unimodular gravity in the next sections.

For a given link L, we shall write for short the quantum observable as $Z(\mathcal{M}; L)$, as defined in Eq.(3.35). We will also choose the action to be that defined by the Chern-Simons theory in $d = 2 + 1$. What Witten achieved in [68] is of significant importance, as it was shown that certain knot polynomials are related to computing correlation functions of these quantum observables, given a suitable Hilbert space structure. Let us start with the simplest case, namely let us consider the three-sphere S^3 as the manifold. This has been shown to be linked with Jones polynomials, first introduced in [36, 37].

In knot theory and Jones theory, one usually makes an analytic continuation and replaces the Chern-Simons level k by a complex variable q, for which special values of q have special properties. These values correspond with the fact that k in Chern-Simons theory has integer values. The two-variable, namely n and k, generalization of the Jones polynomial corresponds to the case where the gauge group is $SU(n)$, with R_i the n-dimensional defining representation of $SU(n)$. We basically analytically continue these variables to allow complex values. Another interesting case is the Kauffman polynomial which arises when taking the gauge group to be $SO(n)$, with R_i the n-dimensional defining representation of $SO(n)$.

The choice of the orientation of the manifold $\mathcal{M}$, as well as the knots also seems to play a key role in defining these knot polynomials. Indeed $Z(\mathcal{M}; L)$ remains invariant under charge conjugation of the representation, which is a consequence of reversing the orientation. These are all features that appear in knot polynomials and thus are required to be preserved for the quantum observable as well.

Connected Sums of Manifolds

The main point is to consider the large k behaviour of the quantum observable, as such a limit corresponds to considering fluctuations about the stationary points of the action; these are the flat connections. With this assumption, consider a 3-manifold $\mathcal{M}$ which is the connected sum of two 3-manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, joined along a two-sphere S^2. Note that there can be knots and links on each 3-manifold, but if so they do not pass through the joining. Now, take any arbitrary 3-manifold $\mathcal{N}$ that may contain knots and links. Then the claim is that the partition function for $\mathcal{N}$ satisfies the relation

$$Z(\mathcal{N}, L)Z(S^3) = Z(\mathcal{M}_1, L_1)Z(\mathcal{M}_2, L_2), \tag{3.36}$$

where $Z(S^3)$ should be understood as the partition function for the three-sphere with no knots and links;

and $L_i,\ i = 1, 2$ are the links in $\mathcal{M}_i$, respectively. Re-writing this for $\mathcal{M}$ gives

$$\frac{Z(\mathcal{M}, L)}{Z(S^3)} = \prod_{i=1}^{2} \frac{Z(\mathcal{M}_i, L_i)}{Z(S^3)}. \tag{3.37}$$

The crucial result is that for $\mathcal{M}_i$ to be copies of S^3 with knots and links, we get results from Jones theory. This result in this context expresses the fact that these invariants are multiplicative when one takes the disjoint sum of knots. To be more precise, $\mathcal{M}_1$ has boundary S^2. We can then associate according to the rules of quantum field theory a physical Hilbert space $\mathcal{H}$ to this boundary, and is one-dimensional. Thus, $Z(\mathcal{M}_1, L_1)$ determines a state or wavefunction ψ in $\mathcal{H}$. Similarly, for $\mathcal{M}_2$, we define an associated dual Hilbert space $\mathcal{H}^*$ to the boundary S^2 of $\mathcal{M}_2$. This is so because this boundary comes with the opposite orientation. Thus we can define a dual wavefunction $\tilde{\psi}$ in $\mathcal{H}^*$ obtained from $Z(\mathcal{M}_2, L_2)$. The connected sum path integral then yields the pairing

$$Z(\mathcal{M}, L) \equiv (\psi, \tilde{\psi}), \tag{3.38}$$

with pairing defined by the mapping $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H}^* \to \mathbb{C}$. We need to know what these states are, which are currently unknown. However, using the fact that these are one-dimensional Hilbert spaces, we can write the states as multiples of states defined on the boundary of S^3. What we mean by that is to take the three-sphere and embed the two-sphere so that it separates S^3 into two three-balls. Then $Z(S^3) = (\phi, \tilde{\phi})$. Using linear algebra facts, we can then write

$$(\psi, \tilde{\psi})(\phi, \tilde{\phi}) = (\psi, \tilde{\phi})(\phi, \tilde{\psi}). \tag{3.39}$$

This indeed reproduces the desired result given by Eq.(3.36). The partition function for the three-sphere is so important that for instance for the case that the gauge group is $SU(2)$, one has

$$Z(S^3) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right). \tag{3.40}$$

This property that the partition function behaves multiplicatively with respect to connected sums can also be used to establish a simple result when $\mathcal{M}$ contains a trivial link. Consider r unlinked and unknotted circles $C_i,\ i = 1, ..., r$ in S^3, each within a representations R_i of the gauge group G. Denoting the partition function of S^3 with this collection of Wilson lines as $Z(S^3; C_1, R_1, ..., C_r, R_r)$, cutting the three-sphere to separate each circles and using the multiplicative formula, we have

$$\frac{Z(S^3; C_1, R_1, ..., C_r, R_r)}{Z(S^3)} = \prod_{i=1}^{r} \frac{Z(S^3; C_i, R_i)}{Z(S^3)^3)}. \tag{3.41}$$

Introducing the normalized expectation value of a link L defined through $\langle L \rangle = Z(S^3; L)/Z(S^3)$, then we obtain

$$\langle C_1 ... C_r \rangle = \prod_i \langle C_i \rangle. \tag{3.42}$$

Focusing on knots in S^3 and gauge group $SU(n)$ with the representation of knots and links in the defining representation, we obtain using skein relation the link expectation values of interest, that is the linear relation

$$\alpha Z(L) + \beta Z(L_1) + \gamma Z(L_2) = 0, \tag{3.43}$$

where L is the original link, and $L_i,\ i = 1, 2,$ are new links obtained by skein relation. This recursion relation for links often appears in knot theory as a linear sum in the ways two crossings occur. The coefficients α, β and γ are weights in the relation. These results are in one-to-one correspondence with the Jones polynomials, and this is the remarkable result achieved in Witten's work. Many possible generalizations have been considered in [68] and we shall not go through them as it is not the main theme of the current thesis, where we will use and employ the terminology developed in the next sections. We would like to stress the importance of the next section, as we will explore the appearance of knots from a completely different physical standpoint, namely in magneto-hydrodynamics (MHD).

3.2 Knots and Relations to Magnetic Helicity

In the previous section, we introduced the concepts of Chern-Simons theory and how it is connected to mathematical results, such as knots and links in knot theory, and how knot polynomials appear by simply studying the quantum field theory of Chern-Simons theory in $2+1$-dimensions. In this section, we would like to show how knots appear in other areas of physics, and especially we will focus on the theory of magneto-hydrodynamics (MHD). This will be of particular importance for the next section, where we will connect the previous and the current section within unimodular gravity and how time has a topological aspect to it in terms of knots and links.

3.2.1 Generalities in Knot Theory

What is a knot exactly? How do we define their twist and linkage among themselves and with other knots? This is going to be about establishing results from knot theory. We will closely follow [24].

The study of invariants of closed curves under smooth deformations is in fact an old problem in mathematics and goes back to the work of Gauss in 1820 with the introduction of the linking number. This is an invariant of two closed curves that measures the number of times one of the curves winds around the other. This is an invariant as the only way to change it is by cutting and gluing the closed curves in a different configuration, resulting in a different value of the linkage. This is also not a smooth deformation, so cannot be allowed. It is given by the following integral:

$$\mathrm{Lk}(C_1, C_2) := \frac{1}{4\pi} \int dx^a \; dy^b \; e_{abc} \frac{(x-y)^c}{|x-y|^3}, \tag{3.44}$$

where C_1 and C_2 are two closed curves. This expression ignores the self-linkage or knottedness of a single closed curve. Note that this expression takes into account the background metric which makes the resulting integral expression diffeomorphism invariant. It has been considered by Maxwell in [50] in the context of electromagnetism, regarding solenoids and the computation regarding magnetic flux lines.

One issue with knot theory is the mathematical classification of knots and links: how can we possibly distinguish two knots (or links) that are smoothly not deformable into each other? This problem has been addressed first within physics with the atomic theory of Maxwell, Kelvin, and Tait [63]. With electromagnetism classically solved by the end of the century, the explanation for atomic spectra was still a big unsolved mystery. The proposed theory contained atoms as knots and had various attractive features such as the stability of atoms with the topological aspect of the knots. The main issue was nevertheless still there, namely, the classification and the mathematical motivation remains to this day, within different physical setups.

Since it is usually hard to visualise in three-dimensional space, projecting the knot into a plane in $\mathbb{R}^2$ is usually more suitable. This however brings extra mathematical complications, and thus new techniques for distinguishing knots from each other are of great importance. Smooth deformations of knots in three-dimensional space translate themselves in a series of motions in terms of the projections and are known as Reidemeister moves. There are three such moves and are depicted in Fig.3.1. These moves in short tell us that if two knot projections can be mapped into each other through a finite number of moves, they are projections of the same knot.

Now coming back to the original motivation in knot theory, namely the classification problem, the hope is to generate a large number of knot invariants. There exists a large number of such invariants, but the problem is still not fully complete. These are generated by appropriate polynomials that we encountered in the previous section, such as the Jones and Kauffman polynomials. These are usually introduced through the braid group, and then skein relations, which we shall briefly mention. Good references include [44, 43, 42].

Braid Group and Knots

Let us introduce the concept of a braid group denoted as B_n. Consider a set of n vertical strings starting and ending in two rows of n horizontally aligned points. These lines can cross each other for a non-fixed amount of number, forming the so-called braid. Now, arranging the lines such that at each horizontal level, there is only one crossing at the ith strand, which we denote g_i, one can describe such a braid by a sequence

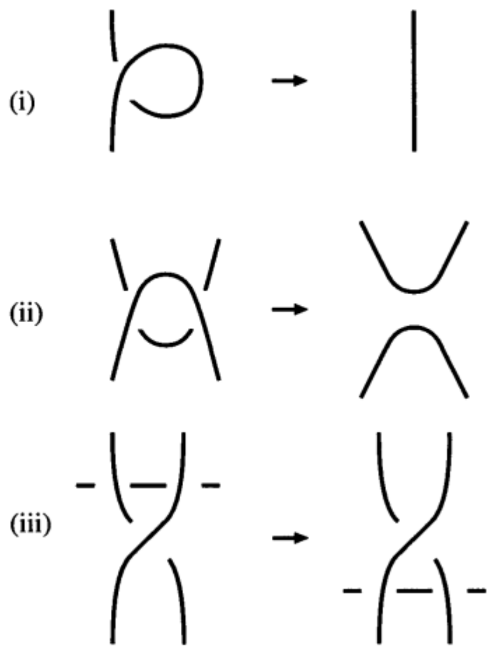

Figure 3.1: Three different Reidemeister moves [24].

$...g_i g_j g_k....$ This ordered sequence means that if one follows the braid from top to bottom and vice-versa, one sees a twist in the lines at the ith and $(i+1)$th-positions followed by a twist of the lines at the jth and $(j+1)$th-positions, and so on. This is depicted in Fig.3.2. Each twist has two possible orientations, denoted by g_i and g_i^{-1}. The twists form a group, the Artin braid group, with group generators given by the twists $g_1, ..., g_{n-1}$ for n lines. The connection with physics here can be seen from identifying the lines as particle

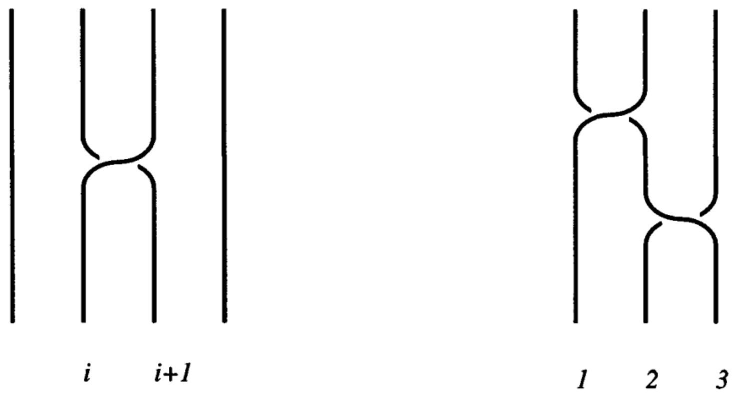

Figure 3.2: Sketch of g_i and $g_1 g_2^{-1}$ [24].

trajectories in $2+1$-dimensional spacetime, which is what we explored in the previous section.

The next question to be asked is how given two braids, what are the conditions for their closure to yield the same knot? Unlike knot diagrams which are described through Reidemeister moves, for braid diagrams these are given by Markov moves. Interestingly, Reidemeister moves of type (iii) are included in the Markov moves, but type (ii) holds up to conjugation under the adjoint action of the braid group: conjugate elements of the braid group are said to be part of type 1 Markov move. The type 2 Markov move is related to type (i) Reidemeister move by identifying a link diagram associated with the closure of a certain braid with its closure. The advantage of the description of knots and links in terms of braids is that one can define link invariants as functionals of the elements of the braid group that are invariant under Markov moves. We can thus introduce representations of the braid group.

Knot Polynomials

What are knot polynomials within the context of knot theory about? A knot polynomial is an assignment of a finite set of numbers to a knot that is invariant under smooth deformations of the knot. Given a knot C, one associates a Laurent polynomial $P(C)_q$ in an arbitrary variable q such that all the coefficients $p_i(C)$ of the polynomial are knot invariants. The first polynomial was introduced by Alexander in [2] which has been updated by Conway in [15], now known in the literature as Alexander-Conway polynomials. These polynomials satisfy the skein relations

$$
\begin{aligned}
P(U)_q &= 1, \\
P(L_+)_q - P(L_-)_q &= qP(L_0)_q,
\end{aligned}
\tag{3.45}
$$

where U is the unknot and $L_\pm$, L_0 are given by the configuration in Fig.3.3. The first relation is a normalization condition for the unknot. The resulting equation from the second relation is a correspondence between the polynomials associated with three different knots. The aim is to apply recursively using the Reidemeister moves until one gets a system of equations for the coefficients with a unique solution. Another knot polynomial has been obtain by Jones [36] and satisfies the skein relations

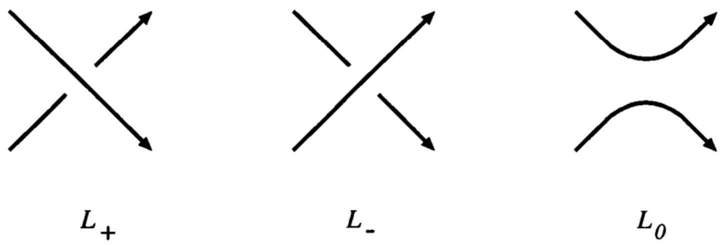

Figure 3.3: Description for the crossings $L_\pm$ and L_0 [24].

$$
\begin{aligned}
J(U)_q &= 1, \\
qJ(L_+)_q - q^{-1}J(L_-)_q &= \left(q^{1/2} - q^{-1/2}\right)J(L_0)_q.
\end{aligned}
\tag{3.46}
$$

Similar constructions do exist that contain both polynomials, such as the HOMFLY polynomials [23]. Notice how we showed in the previous section that topological quantum field theory can be used to obtain these and many more knot polynomials, where specific values for q were obtained given a gauge group and Chern-Simons level k. Despite such great results coming from physics and mathematics, no polynomial known at present is sufficient to distinguish all knots and thus remains an open problem.

Let us for completeness focus on type (i) Reidemeister move. Kauffman introduced a new polynomial that extended the work of Jones to include the additions of curls in the knots:

$$
\begin{aligned}
K(U)_q &= 1, \\
q^{1/4}K(L_+)_q - q^{-1/4}K(L_-)_q &= \left(q^{1/2} - q^{-1/2}\right)K(L_0)_q, \\
K(\hat{L}_\pm)_q &= q^{\pm 3/4}K(\hat{L}_0)_q,
\end{aligned}
\tag{3.47}
$$

where $\hat{L}_\pm$ and $\hat{L}_0$ are given in Fig.3.4. Such invariants are defined for the case of regular isotopy of curves. These can be associated with ambient isotopic invariants of oriented ribbons if one gives a prescription to correspond a ribbon to each curve. These are called framings: we assign a vector to each point of the curve such that one obtains a second curve by infinitesimally shifting the original one along the vector.

Writhe, Twist, and Self-Linkage

With the notion of framing defined above, we can now define the three most important concepts regarding knots and links, that is writhe, twist, and self-linkage.

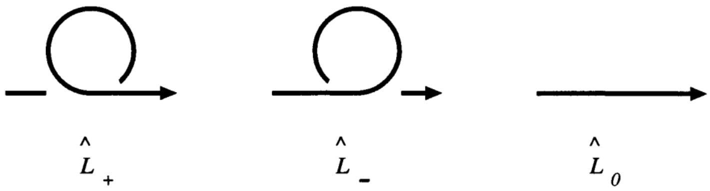

Figure 3.4: Crossing for the skein relations of regular isotopic invariants [24].

The writhe of a knot diagram is given by

$$\mathrm{Wr}(C) := \sum_{\text{crossings}} \epsilon(\text{crossing}), \quad \epsilon(L_\pm) = \pm 1. \tag{3.48}$$

This measures the number of curls in a given knot diagram and is not invariant under type (i) Reidemeister move. It is nevertheless a regular isotopic invariant.

Another regular isotopy invariant is the twist not of a knot, but rather of a band. If both sides of the band are different in say color, then the twist measures how many times the color changes as seen from the planar projection. Performing a type (i) move, one can exchange a twist in a band by a curl.

The self-linking number can be given a definition in terms of writhe and twists: using the fact that in bands type (i) moves exchange curls and twists, one can combine the previous two quantities and obtain a quantity associated with the knot diagram through a framing. This has been given by White's theorem as

$$\mathrm{SLk}(C) := \mathrm{Wr}(C) + \mathrm{Tw}(C), \tag{3.49}$$

where $\mathrm{Tw}(C)$ is the twist of C after framing. The Gauss linking number can also be made algebraic by performing the planar projection technique of the knot from $\mathbb{R}^3$ to $\mathbb{R}^2$:

$$\mathrm{Lk}(C_1, C_2) := \frac{1}{2} \sum_{\text{crossings}(C_1, C_2)} \epsilon(\text{crossing}). \tag{3.50}$$

With the definitions of knot theory coming to an end, let us now explore the realm of MHD that will be relevant in understanding the next section.

3.2.2 Magneto-Hydrodynamics and Helicity

For this subsection, we will describe the main results useful for the next section within the MHD theory. We will explain what magnetic helicity is and how it relates to knot theory. We will mainly follow the widely known papers [7, 8, 52, 51].

Magnetic Helicity and Topology of Flux Tubes

Magnetic fields $\mathbf{B}$ have zero divergence, i.e. they satisfy Gauss's law for magnetism $\nabla \cdot \mathbf{B} = 0$. This makes magnetic field lines have no endpoints. This allows us to examine the topological structures of these fields in terms of the topology of closed curves, which correspond to the study of knots and links introduced previously.

Consider the following integral for a given vector field $\mathbf{V}$

$$I = \int_\Sigma d^3x \, \mathbf{V} \cdot (\nabla \times \mathbf{V}), \tag{3.51}$$

where Σ is some constant time spacelike hypersurface. It has been shown by Moffatt et al. that this integral can be associated with the topological properties of the field lines of the curl of the vector field $\mathbf{V}$. The canonical example that will be of tremendous importance for the next section is magnetic helicity integral defined as

$$H := \int_\Sigma d^3x\ \mathbf{A} \cdot \nabla \times \mathbf{A} \equiv \int_\Sigma d^3x\ \mathbf{A} \cdot \mathbf{B}, \tag{3.52}$$

where $\mathbf{B} := \nabla \times \mathbf{A}$, and $\mathbf{A}$ is the magnetic vector potential. Magnetic helicity measures the linkage of magnetic field lines; for two untwisted closed flux tubes linked once, one has

$$H = \pm 2\Phi_1\Phi_2, \tag{3.53}$$

where $\Phi_i,\ i = 1, 2$ is the strength of the magnetic flux tubes. This quantity also relates to topological constructions such as the Hopf invariant and the Gauss linking number as introduced in the previous subsection. Importance should be given to the fact that the integration is defined over all of the hypersurface with fields vanishing at infinity: the boundary $\partial\Sigma$ is a magnetic surface, that is $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$. We thus want to have no linkage at the boundary of this volume.

Gauge-Invariance and Conservation of Helicity

Consider now a gauge transformation of the magnetic vector potential $A \mapsto A + \nabla\chi$. Since the magnetic field is gauge-invariant by construction of the curl of the magnetic vector potential, the magnetic helicity integral transforms according to

$$\begin{aligned} H \longmapsto& \int_\Sigma d^3x\ (\mathbf{A} + \nabla\chi) \cdot \mathbf{B} \\ &= H + \int_\Sigma d^3x\ \nabla\chi \cdot \mathbf{B} \\ &= H + \int_\Sigma d^3x\ \nabla \cdot (\chi\mathbf{B}) - \int_\Sigma d^3x\ \chi \underbrace{(\nabla \cdot \mathbf{B})}_{=0} \\ &= H + \int_{\partial\Sigma} d^2x\ \chi\mathbf{B} \cdot \hat{\mathbf{n}}. \end{aligned} \tag{3.54}$$

Now the second integral vanishes if the boundary of Σ is a magnetic surface. Thus, magnetic helicity is gauge-invariant if and only if at the boundary all fields go to zero. Also, notice that we have assumed a simply connected volume containing Σ when performing the second integration, as for other geometries, the gauge parameter χ may be multi-valued, which is a problem.

Another useful property of magnetic helicity is that it may not necessarily be conserved when finite resistivity η is present. Assuming a linear Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta\mathbf{J}, \tag{3.55}$$

where $\mathbf{E}$ is the electric field, $\mathbf{v}$ is the velocity vector, and $\mathbf{J}$ is the magnetic current density. Using the standard definitions for electric and magnetic fields, and the Maxwell-Faraday equation, i.e.

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \\ \frac{\partial\mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}, \end{aligned} \tag{3.56}$$

we can evaluate the time derivative of the magnetic helicity expression. Now we will assume that the field lines are co-moving with the hypersurface Σ through time, a very important point for later results. This enables us to exchange total derivatives with partial derivatives when passing inside the integrals:

$$\begin{aligned} \frac{dH}{dt} &= \int_{\Sigma(t)} d^3x \left(\frac{\partial\mathbf{A}}{\partial t} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{\partial\mathbf{B}}{\partial t} \right) \\ &= \int_{\Sigma(t)} d^3x\ (-\mathbf{E} - \nabla\phi) \cdot \mathbf{B} - \int_{\Sigma(t)} d^3x\ \mathbf{A} \cdot (\nabla \times \mathbf{E}) \\ &= -2\int_{\Sigma(t)} d^3x\ \mathbf{E} \cdot \mathbf{B}, \end{aligned} \tag{3.57}$$

where in the last step, we performed an integration by parts on $\mathbf{A} \cdot (\nabla \times \mathbf{E})$ and assumed fields vanish at infinity. We also used a gauge fixing of the electric scalar potential $\phi = 0$. Using the above-mentioned Ohm's law, we have the result

$$\frac{dH}{dt} = -2 \int_{\Sigma(t)} d^3x \; \eta \mathbf{J} \cdot \mathbf{B}. \tag{3.58}$$

The simplest form of MHD is the ideal MHD, which assumes that the resistivity is very small and thus is taken to be zero. Then it can be seen that magnetic helicity is conserved. From Eq.(3.57), it can be seen that for a zero velocity field $\mathbf{v} = \mathbf{0}$, the electric field vanishes. The non-conservation of helicity is very important for the next section, so keep an eye on this very closely!

Helicity Decomposition and Gauss Linking Number

We want to now understand how to compute for general setups the helicity integral. It is often useful to separate space into regions bounded by magnetic surfaces. The magnetic helicity integral of the associated field can then be decomposed into a sum of internal and external helicities. The internal component contributes to the structure inside each region, while the external component contributes to the interlinkages among the regions. This has been computed explicitly in [7] and we will give a short explanation here.

Consider an arbitrarily twisted, kinked, and knotted flux tube. We can then express the helicity as a sum of twist and kink helicities:

$$H = H_T + H_K. \tag{3.59}$$

The external helicity of a knot is defined to be equal to H_K if the knot exhibits a minimum of crossovers. For two linked flux tubes, the external helicity is given in terms of the Gauss linking number $2\mathrm{Lk}(C_1, C_2)\Phi_1\Phi_2$. For larger values of linking, internal contributions must also be taken into account as the tubes become more kinked. However, this can still be ignored in a particular case. Take a large number N of closed flux elements, each containing a small flux $\delta\Phi$. Then the helicity computation reads

$$H = \sum_{i,j=1}^{N} \mathrm{Lk}(C_i, C_j)(\delta\Phi)^2 + \sum_{i=1}^{N} H_{i,\text{ int}}. \tag{3.60}$$

In this expression, $(\delta\Phi)^2$ and $H_{i,\text{ int}}$ vary as N^{-2}. As $N \to \infty$ the interlinkage sum will reach a finite limit since it contains N^2 terms. but the internal helicity sum vanishes as N^{-1}. Notice how Eq.(3.59) and Eq.(3.49) are related to each other.

With all these facts on knots, Chern-Simons theory, and magnetic helicity cleared out, we are a present ready to introduce the topological aspect of unimodular time within unimodular gravity, which is the most interesting part of the thesis.

3.3 Vorticity, Topology and Time

In this section, we will go back to unimodular gravitation and perform some basic modifications for the unimodular action. The main motivation for doing so relies on the fact that the theory has topological properties locally in terms of propagation: there are no propagating degrees of freedom for the vector density $\mathcal{T}^\mu$. We are thus naturally led to ask the question: what other possible topological contributions give rise to similar results? The first clue is the gauge invariance of the unimodular action under transformations $\mathcal{T}^\mu \mapsto \mathcal{T}^\mu + \epsilon^\mu$ with $\partial_\mu \epsilon^\mu = 0$. The main point of this section is to assign a gauge group for this residual gauge transformation. It turns out that taking $U(1)$ gauge group results in a similar theory to that of the abelian Chern-Simons theory in $3+1$-dimensions with extra features. This will result in identifying and explicitly showing an expression for what the unimodular time T really corresponds to, and will be shown to give rise to magnetic helicity introduced previously. This will be given a possible physical interpretation in terms of spacetime volume creation and how time flow can really be created with knots and magnetic field lines within unimodular gravity theories. Finally, we will extend the model to non-abelian gauge groups and show how magnetic helicities of different magnetic field lines interact among themselves and give rise to unimodular times with different flavors. The results here are currently under study for further work and will appear in a paper shortly [22].

3.3.1 U(1)-Symmetry and Vortices

Consider for starters the case of taking the cosmological constant as a variable and where the gravitational Newton coupling is normalized to $8\pi G = 1$. We want to construct a theory that is topological in terms of local propagation and has a similar symmetry and invariance as standard HT-gravity.

The New Action and Field Equations

For this consider the $U(1)$ Pontryagin term for the unimodular term:

$$S_{\mathrm{U}} = \int d^4x \, \frac{\Lambda}{4} \mathcal{F}^{\mu\nu} F_{\mu\nu}. \tag{3.61}$$

The $U(1)$-field strength two-form $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ has a dual field strength given by

$$\mathcal{F}^{\mu\nu} := \frac{1}{2} F_{\alpha\beta} e^{\alpha\beta\mu\nu} \equiv (\partial_\alpha A_\beta) \, e^{\alpha\beta\mu\nu}. \tag{3.62}$$

We can then write the contraction of the dual field strength with the field strength itself as

$$\begin{aligned}
\mathcal{F}^{\mu\nu} F_{\mu\nu} &= \frac{1}{2} F_{\alpha\beta} \, e^{\alpha\beta\mu\nu} \, F_{\mu\nu} \\
&= 2(\partial_\alpha A_\beta) \, e^{\alpha\beta\mu\nu} \, \partial_\mu A_\nu \\
&= 2\partial_\alpha (e^{\alpha\beta\mu\nu} A_\beta \partial_\mu A_\nu) - 2A_\beta \underbrace{\partial_\alpha (e^{\alpha\beta\mu\nu} \partial_\mu A_\nu)}_{e^{\alpha\beta\mu\nu}\partial_{\alpha\mu} A_\nu = 0} \\
&= 2\partial_\alpha (A_\beta \mathcal{F}^{\alpha\beta}).
\end{aligned} \tag{3.63}$$

Thus, the new unimodular term reads

$$S_{\mathrm{U}} = \int d^4x \, \frac{\Lambda}{2} \partial_\alpha (A_\beta \mathcal{F}^{\alpha\beta}). \tag{3.64}$$

This looks very similar to the standard HT gravity term if one defines the vector density $\mathcal{V}^\alpha := \frac{1}{2} A_\beta \mathcal{F}^{\alpha\beta}$. The action can thus be re-written in the familiar form

$$S_{\mathrm{U}} = \int d^4x \, \Lambda \partial_\alpha \mathcal{V}^\alpha. \tag{3.65}$$

The full action then reads

$$S = \frac{1}{2} \int d^4x \, \sqrt{-g} \, (R - 2\Lambda) + \int d^4x \, \Lambda \partial_\alpha \mathcal{V}^\alpha. \tag{3.66}$$

Variation with respect to Λ gives

$$\frac{\delta S}{\delta \Lambda} = 0 \quad \Longleftrightarrow \quad \sqrt{-g} = \partial_\alpha \mathcal{V}^\alpha. \tag{3.67}$$

Varying with the metric leads to the same variable Einstein equations as Eq.(2.29). A more subtle point appears upon varying the action with the gauge fields A_μ:

$$\frac{\delta S}{\delta A_\nu} = 0 \quad \Longleftrightarrow \quad \mathcal{F}^{\mu\nu} \partial_\mu \Lambda = 0. \tag{3.68}$$

This can be seen in a very straightforward way. The variation gives

$$\begin{aligned}
0 = \frac{\delta S_{\mathrm{U}}}{\delta A_\nu} &= \frac{\delta \mathcal{L}_{\mathrm{U}}}{\delta A_\nu} \\
&= \frac{\partial \mathcal{L}_{\mathrm{U}}}{A_\nu} - \partial_\mu \frac{\partial \mathcal{L}_{\mathrm{U}}}{\partial(\partial_\mu A_\nu)}.
\end{aligned} \tag{3.69}$$

The first term vanishes while the second term gives

$$
\begin{aligned}
\frac{\partial \mathcal{L}_{\mathrm{U}}}{\partial(\partial_\lambda A_\sigma)} &= \frac{\Lambda}{2}\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\lambda A_\sigma)}(\partial_\alpha A_\beta)e^{\alpha\beta\mu\nu} + \frac{\Lambda}{2}(\partial_\mu A_\nu)\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\lambda A_\sigma)}e^{\alpha\beta\mu\nu} \\
&= \Lambda\delta^\lambda_{[\mu}\delta^\sigma_{\nu]}(\partial_\alpha A_\beta)e^{\alpha\beta\mu\nu} \\
&= \Lambda(\partial_\alpha A_\beta)e^{\lambda\sigma\alpha\beta}.
\end{aligned} \tag{3.70}
$$

Thus, taking the divergence gives the desired result:

$$
\begin{aligned}
\partial_\mu \frac{\partial \mathcal{L}_{\mathrm{U}}}{\partial(\partial_\mu A_\nu)} &= \partial_\mu(\Lambda\partial_\alpha A_\beta)e^{\alpha\beta\mu\nu} \\
&= \mathcal{F}^{\mu\nu}\partial_\mu\Lambda.
\end{aligned} \tag{3.71}
$$

In [5], it has been shown that the most general solution to such an equation, assuming that $d\Lambda \neq 0$ is

$$
A = da + bd\Lambda, \quad a, b \in \mathcal{C}^\infty(\mathcal{M}). \tag{3.72}
$$

An important consequence of this solution involves having a good choice of global coordinate Λ: the level surfaces of constant Λ provide a globally nondegenerate foliation of $\mathbb{R}^4$, with each slice having topology $\mathbb{R}^3$. However, we can also have solutions to Eq.(3.68) using the invertibility of the dual-field strength, which makes $\Lambda =$ const. trivially true. Thus, it seems that there exists a subclass of solutions within the $F \wedge F$ approach to unimodular gravity. The non-constant Λ case will be further studied in the upcoming paper [22].

This type of extension to HT-gravity has been considered prior to this work in the literature. For instance, see [34, 35, 33, 25]. The difference in what we have is a meaning to the unimodular time within such a framework which is of tremendous importance that has not been fully considered in the previous works.

Let us decompose $\mathcal{F}^{\mu\nu}F_{\mu\nu}$ in $3+1$:

$$
\begin{aligned}
\frac{1}{4}\mathcal{F}^{\mu\nu}F_{\mu\nu} &= \frac{1}{4}(\mathcal{F}^{0\nu}F_{0\nu} + \mathcal{F}^{j\nu}F_{j\nu}) \\
&= \frac{1}{2}\mathcal{F}^{0i}F_{0i} + \frac{1}{4}m\mathcal{F}^{jk}F_{jk}.
\end{aligned} \tag{3.73}
$$

Let us define the following quantities for later convenience:

$$
F_{0i} = E_i, \quad \text{and} \quad \mathcal{F}^{0i} = \mathcal{B}^i, \tag{3.74}
$$

where E_i can be thought of as the electric field and $\mathcal{B}^i$ as a densities magnetic field. Then, we can safely rewrite

$$
\begin{aligned}
\frac{1}{2}\mathcal{F}^{0i}F_{0i} &= \frac{1}{2}\mathcal{B}^iE_i. \\
\frac{1}{4}\mathcal{F}^{jk}F_{jk} &= \frac{1}{8}F_{\alpha\beta}e^{\alpha\beta jk}F_jk \\
&= \frac{1}{8}F_{0i}e^{0ijk}F_{jk} + \frac{1}{8}F_{i0}e^{i0jk}F_{jk} \\
&= \frac{1}{4}F_{0i}e^{0ijk}F_{jk} \\
&= \frac{1}{2}\mathcal{F}^{0i}F_{0i} \equiv \frac{1}{2}\mathcal{B}^iE_i.
\end{aligned} \tag{3.75}
$$

Thus, the $3+1$ splitting gives the compact expression

$$
\frac{1}{4}\mathcal{F}^{\alpha\beta}F_{\alpha\beta} = E_i\mathcal{B}^i, \tag{3.76}
$$

and the unimodular term can be written as the topological term

$$S_{\mathrm{U}} = \int d^4x \; \Lambda E_i \mathcal{B}^i. \tag{3.77}$$

Such terms are also called axionic, where Λ behaves here as an axion.

Magnetic helicity as Time and Non-Conservation

In similar lines with unimodular gravity, one can introduce the gauge fixing $\mathcal{V}^i = 0$. With this, one has the $\mathcal{V}^0$ contribution that remains non-zero and is given by

$$\begin{aligned} \mathcal{V}^0 &= \frac{1}{2} A_\beta \mathcal{F}^{0\beta} \\ &= \frac{1}{2} A_i \mathcal{F}^{0i} \equiv \frac{1}{2} A_i \mathcal{B}^i. \end{aligned} \tag{3.78}$$

Under the Coulomb gauge fixing $A_0 = 0$, we would have for the time derivative of the magnetic field

$$\begin{aligned} \dot{\mathcal{B}}^i &= e^{0ijk} \partial_j \dot{A}_k \\ &= -e^{0ijk} \partial_j E_k \equiv e^{ijk} \partial_j E_k, \end{aligned} \tag{3.79}$$

where with the gauge fixing, $E_i = \dot{A}_i$. We can then write the time derivative of $\mathcal{V}^0$ as:

$$\begin{aligned} \dot{\mathcal{V}}^0 &= \frac{1}{2}\left(\dot{A}_i \mathcal{B}^i + A_i \dot{\mathcal{B}}^i \right) \\ &= \frac{1}{2}\left(E_i \mathcal{B}^i + E_i \mathcal{B}^i \right) + \text{b.t.} \\ &\equiv -E_i \mathcal{B}^i + \text{b.t.} \end{aligned} \tag{3.80}$$

This expression is an important result regarding the temporal dependence of the density $\mathcal{V}^0$. The result we have is that it is not a constant in coordinate time unless we do not have any electric field, which in the chosen gauge implies that A_i is time-independent, which is not the case in general settings as we shall delve more shortly.

Similar to HT-gravity, we can introduce an unimodular time $T(t)$ as

$$T(t) := \int_{\Sigma(t)} d^3x \; \mathcal{V}^0 \equiv \frac{1}{2} \int_{\Sigma(t)} d^3x \; A_i \mathcal{B}^i. \tag{3.81}$$

The right-hand side of this four-volume time is what we have introduced in the previous section, namely, magnetic helicity:

$$\begin{aligned} H(t) &:= \int_{\Sigma(t)} d^3x \; A_i \mathcal{B}^i \\ &\equiv \int_{\Sigma(t)} d^3x \; \mathbf{A} \cdot \mathbf{B}, \end{aligned} \tag{3.82}$$

where here the dot-product is over some specified background metric on the hypersurface $\Sigma(t)$ at a given constant coordinate time t. This quantity as discussed previously is widely used in MHD as a quantitative measure of how vortex lines link together. The remarkable feature observed in the context of the present situation is that this quantity appears to describe the four-volume time. Notice that the quantity is gauge-invariant under $U(1)$ gauge transformations only if we consider the boundary of the hypersurface to be a magnetic surface, i.e. $\mathcal{B}^i \hat{n}_i = 0$, for any unit normal vector $\hat{n}_i$. This condition is equivalent to assuming all fields vanish at spacial infinity or boundary $\partial\Sigma$ when performing integration by parts, also discussed previously.

What is the physical meaning of magnetic helicity as an appropriate physical time? According to the first equality in Eq.(2.43), we have a relation between the four-volume time difference and the four-volume between initial and final coordinate times. Replacing the time with the expression for magnetic helicity given by Eq.(3.82) one has a rather interesting connection to the four-volume

$$V(t_i \to t_f) = -\frac{1}{2}\Delta H = \Delta T. \tag{3.83}$$

Magnetic helicity is a measure of the linkage of magnetic vortex lines [51]. Connecting this fact with the four-volume means the evolution of the knottedness and the linkage among these magnetic vortices through each hypersurface gives the spacetime four-volume. This result can in turn be better understood from the standpoint of the creation of spacetime volume via the change of linkage of the links with coordinate time.

As previously argued, it has been shown in [7] that upon dividing the larger spacial volume into a collection of sub-volumes each containing single magnetic flux tubes, magnetic helicity can be decomposed into internal and external helicities, with the former arising from internal structure within each flux tubes, and the latter arising from interactions among each flux tubes, also known as linking and knotting. Mathematically this is expressed for m-tubes

$$H = \sum_{I,J=1}^{m} \kappa_{IJ}\Phi_I\Phi_J + + \sum_{I=1}^{m} H_{I,\text{int}}, \tag{3.84}$$

where $\kappa_{IJ} \in \mathrm{Z}$ and Φ_I is the flux tubes of each vortex line $I, J = 1, ..., m$. In the large-N limit, the internal helicity sum vanishes, while the external helicity remains finite. This makes the connection between the number of linkages to four-volume time and volume apparent. This result has also been extended to the helicity of open (continuous) field structures, by dividing the spacial volume into simply connected regions. We thus have an expression of the Gauss linking integral formula in terms of κ_{IJ}. Within the abelian version of the framework laid out in Witten's paper, one can identify the Chern-Simons number with the magnetic helicity. We can thus also use the abelian Chern-Simons number to interpret the unimodular time. We will expand this further shortly to include the non-abelian case as well.

A rather important aspect of magnetic helicity in the literature is that it is a conserved quantity in time, yet is not valid anymore within the unimodular gravity context. This can be seen by taking the time derivative of Eq.(3.82) while assuming that the magnetic field lines are co-moving with the hypersurface, resulting in

$$\begin{aligned}\dot{T} &= -\frac{1}{2}\dot{H} \\ &= \int_{\Sigma(t)} d^3x\, E_i\mathcal{B}^i \neq 0.\end{aligned} \tag{3.85}$$

In addition, on-shell one has

$$\dot{T} = \int_{\Sigma(t)} d^3x\, N\sqrt{h} \neq 0, \tag{3.86}$$

with N the lapse function and h the determinant of the induced metric on the hypersurface. Combining these equations leads to

$$\int_{\Sigma(t)} d^3x\, E_i\mathcal{B}^i = \int_{\Sigma(t)} d^3x\, N\sqrt{h}. \tag{3.87}$$

What is really going on physically here? One possibility that is highly unlikely to happen is that by assuming the magnetic field lines in a resistive medium of resistivity ρ and satisfying a linear Ohm's law

$$E_i = \rho J_i, \tag{3.88}$$

with J_i the electric current density. We can then define the helicity-dissipation rate as analogous to Eq.(3.85):

$$\dot{T} = \int_{\Sigma(t)} d^3x\, \rho J_i\mathcal{B}^i. \tag{3.89}$$

The right-hand side of this equation is related (on-shell) to the induced metric, we have the resistive medium as the hypersurface itself. However, this implies the existence of a current density, which we do not have within unimodular gravity since the theory is purely topological, implying the vanishing of the electric field, and thus the conservation of unimodular time, a result that we do not want to consider, as it simply means that there are no four-volume creation. Further investigation regarding the implications of this has currently been made within the upcoming paper.

Gauge transformations and the scalar term

At first sight, the theory looks identical to unimodular gravity, yet subtleties are present. One such thing is due to gauge transformations of the $U(1)$ gauge field

$$A_\mu \longmapsto A_\mu + \partial_\mu \chi, \quad \chi \in C^\infty(\mathcal{M}), \tag{3.90}$$

leaving the field strength and its dual invariant. However the vector density $\mathcal{V}^\alpha$ transforms according to

$$\mathcal{V}^\alpha \longmapsto \mathcal{V}^\alpha + \frac{1}{2}(\partial_\beta \chi)\mathcal{F}^{\alpha\beta} = \mathcal{V}^\alpha + \text{b.t.}, \tag{3.91}$$

i.e. invariant up to a boundary term, and where we used the Bianchi identity $\partial_\beta \mathcal{F}^{\alpha\beta} = 0$. This makes Eq.(3.65) invariant under general $U(1)$ gauge transformations. Note the ϵ^α-parametrization of the gauge transformation as in $\mathcal{V}^\mu \mapsto \mathcal{V}^\mu + \epsilon^\mu$ is given for this case by the boundary term mentioned above, that is :

$$\epsilon^\alpha = \partial_\beta \left(\frac{1}{2}\chi \mathcal{F}^{\alpha\beta}\right), \tag{3.92}$$

with spacetime components

$$\begin{aligned} \epsilon^0 &= \partial_i \left(\frac{1}{2}\chi \mathcal{B}^i\right), \\ \epsilon^i &= -\partial_0 \left(\frac{1}{2}\chi \mathcal{B}^i\right). \end{aligned} \tag{3.93}$$

In [27], a similar parametrization appears through the use of constraints. A connection clearly exists between these theories by setting the gauge parameter $\alpha^i = -\frac{1}{2}\chi\mathcal{B}^i$. This parametrization in turn leads to the transverse condition $\partial_\alpha \epsilon^\alpha$, i.e. $\epsilon^0 = -\partial_i \alpha^i$ and $\epsilon^i = \dot{\alpha}^i$. Note that in the standard HT-gravity, we have three independent gauge modes, while in this new setup, we only have one, due to the infinitesimal $U(1)$ gauge parameter χ. This subtlety in the gauge modes is a key feature that makes both theories not fully equivalent, and indeed this new theory is a subclass of unimodular gravity theories equivalent to the HT-gravity classically. This might have connections with the Proca term as discussed at the end of the previous chapter. This link with Proca theory and its quantum aspects will be studied in future work.

Towards a theory of many-helicities

We would like to now extend from single variable to multi-variable constants $\boldsymbol{\alpha}$ as in Sec.2.3. This introduces within the new framework of helicity a multitude of magnetic fields $\mathcal{B}^i_{\boldsymbol{\alpha}}$. For every gauge field $A^{\boldsymbol{\alpha}}_\mu$, one considers the action

$$S_{\text{U}} = \int d^4x\, \boldsymbol{\alpha} \cdot \partial_\mu \mathcal{V}^\mu_{\boldsymbol{\alpha}}, \tag{3.94}$$

where the vector densities $\mathcal{V}^\mu_{\boldsymbol{\alpha}}$ given by $\mathcal{V}^\mu_{\boldsymbol{\alpha}} = \frac{1}{2}A^{\boldsymbol{\alpha}}_\nu \mathcal{F}^{\mu\nu}_{\boldsymbol{\alpha}}$. This in turn gives $\boldsymbol{\alpha}$-many magnetic helicities defined analogously to Eq.(3.82). We are thus expecting to get similar results to Eq.(2.74) in terms of physical time differences. The helicity approach to the extended unimodular gravity thus provides a new interpretation for the observed values of various constants. For instance, the observed value of the cosmological constant in Eq.(2.82) gets replaced with the ratio of the difference in magnetic helicities of Ricci- and four-volumes, respectively. In the large vortex limit, we have a ratio of integer difference multiplied by a ratio of the difference in magnetic fluxes for each magnetic field $\mathcal{B}^i_{\alpha_{1,2}}$. This perspective can thus shed new light on the cosmological constant problem and its observed value. There are no apparent reasons why this does not apply to other base theories that differ from standard general relativity, such as Brans-Dicke or higher-derivative theories.

There is a much nicer way of viewing these variables $\boldsymbol{\alpha}$. Consider a diagonal matrix K_{IJ}, with $I, J = 1, ..., d$, with d the dimension of the vector space formed by each variable $\boldsymbol{\alpha}$. It is given by $K_{IJ} = diag(\alpha_1, \alpha_2, ..., \alpha_d)$. We can then define a Chern-Simons theory with gauge group the d-torus with action given by

$$S_{\text{U}} = -\frac{1}{2}\int K_{IJ}F^I \wedge F^J. \tag{3.95}$$

The action has symmetry group $U(1)^d$, the torus group. This can be associated with the gauge transformation of $\mathcal{T}_{\boldsymbol{\alpha}}$ without any loss of generality. Thus each unimodular-like time comes with its gauge field A^I, as required from the discussion above, and hence an associated magnetic helicity.

The next section is devoted to pushing the model to non-abelian territories and looking for further topological aspects of unimodular time.

3.3.2 Topology of Time

We will now extend the model to non-abelian gauge theories and explore the deeper connections of topology with unimodular time T, such as understanding the winding number, Wilson loops and knots.

Non-abelian extension

We want to formulate a non-abelian unimodular term that is fully equivalent to the standard HT-gravity discussed previously. Such an action is described by the unimodular-like term

$$S_{\text{U}} = -\frac{1}{2}\int \Lambda Tr[F \wedge F], \tag{3.96}$$

where the trace map is on a representation of the Lie group G underlying the theory. The non-abelian extension of the field strength tensor is the curvature two form given by $F = dA + A \wedge A$, with Lie algebra-valued gauge fields $A = A^a T^a$ given in terms of generators $\{T^a\}_a$ satisfying the algebra $[T^a, T^b] = f^{abc}T^c$ and trace rule $Tr[T^a T^b] = \delta^{ab}/2$. One can use the explicit definition of the field strength to recast the action into a total derivative term

$$S_{\text{U}} = \int \Lambda dV, \tag{3.97}$$

where $V \in \Omega^3(\mathcal{M})$ is given by the well-known Chern-Simons three form:

$$V := -Tr\Big[A \wedge dA + \frac{2}{3}A \wedge A \wedge A\Big]. \tag{3.98}$$

The one-form dual $\mathcal{V}^\alpha$ has thus spacetime components given by

$$\mathcal{V}^\alpha = e^{\alpha\beta\mu\nu} Tr\Big[A_\beta \partial_\mu A_\nu + \frac{2}{3}A_\beta A_\mu A_\nu\Big]. \tag{3.99}$$

Introducing once more the unimodular gauge fixing condition $\mathcal{V}^i = 0$ and the unimodular time given by the spacial flux of $\mathcal{V}^0$ gives the well-known Chern-Simons density, namely:

$$\mathcal{V}^0 = -\frac{1}{2}e^{ijk}\left(A_i^a \partial_j A_k^a + \frac{1}{3}f^{abc}A_i^a A_j^b A_k^c\right). \tag{3.100}$$

This density can then be integrated over each constant time hypersurface $\Sigma(t)$ to give an explicit expression for the four-volume time, that is

$$\begin{aligned} T(t) &:= -\frac{1}{2}\int_{\Sigma(t)} d^3x\, e^{ijk}\left(A_i^a \partial_j A_k^a + \frac{1}{3}f^{abc}A_i^a A_j^b A_k^c\right) \\ &\equiv -\frac{1}{2}Y_{\text{CS}}(t). \end{aligned} \tag{3.101}$$

The term $Y_{\text{CS}}(t)$ is the Chern-Simons number (ignoring the prefactors appearing in this number). Using the unimodular condition adapted for the non-abelian case, one can show that

$$\sqrt{-g} = -\frac{1}{2}Tr[F \wedge F] \implies \Delta T = -\frac{1}{2}\Delta Y_{\text{CS}} = V(t_i \to t_f), \tag{3.102}$$

which is equivalent to Eq.(3.83) with the magnetic helicity of $U(1)$ replaced with the Chern-Simons number. Notice that the first term is a sum of helicities $H^a(t)$ running over $a = 1, ..., \dim(G)$, with the helicity not necessarily being magnetic, but rather a more general topological notion. The second term is a mixing term, putting together the various vortex lines, and making it possible to allow interactions among various Lie algebra-valued vortex lines.

We would like to discuss the situation under which a non-abelian gauge transformation is applied to the unimodular time. It is well known that for such gauge transformations, the winding number appears for the Chern-Simons number transformation. To be more precise, the Chern-Simons number is invariant under elements of the connected components of the gauge group. However, taking gauge transformations associated with non-zero elements of the homotopy group $\pi_3(G)$, there is no such invariance. Thus the Chern-Simons number is not invariant under gauge transformations of the non-zero winding number, that is

$$Y_{\text{CS}} \longmapsto Y_{\text{CS}} + w(g), \tag{3.103}$$

where $w(g)$ is the winding number of the gauge group element g of the homotopy group. It is given by the Chern-Simons number over a three-manifold $\mathcal{X}$ of a pure gauge configuration:

$$w(g) := -\frac{1}{3}\int_{\mathcal{X}} Tr\Big[\big(gdg^{-1}\big)^3\Big]. \tag{3.104}$$

For simple and compact Lie group G, the third-homotopy group coincides with the set of integer numbers, i.e. $\pi_3(G) \cong \mathbb{Z}$. Thus the winding number is labelled by integers and counts the amount of wrapping of the three-manifold around the gauge group.

Applying this to the unimodular time, under a gauge transformation of the unimodular time for non-abelian gauge fields, one has an integer label for the wrapping of the constant-time hypersurface around a simple and compact gauge group G. We thus see that the non-abelian unimodular time of a non-abelian pure gauge is proportional to the winding number. For instance, we can take the gauge group to be $SU(2)$ with three generators and the three-manifold to be S^3. Then the third homotopy group is the set of equivalence classes of continuous maps among three spheres $S^3 \to S^3$. The winding number measures the amount of times the three-sphere wraps around itself. One could naturally consider the maps $S^3 \to S^2$ which gives the Hopf fibration, but since the two-sphere is not a gauge group, we are not considering such maps. The case of having a pure gauge setup is not adapted to our situation, as this corresponds to a solution with field strength $F = 0$. This would require the unimodular condition $V = 0$, i.e. we have no spacetime volume, a contradiction. Thus we cannot consider such solutions as adequate.

Wilson loops and unimodular time

We wish to pick a suitable set of gauge invariant local operators to describe our unimodular time. Considering a principal G-bundle, with base space Σ, the constant-time hypersurface. Let γ be a curve on Σ between two points $p, q \in \Sigma$ and A_i the gauge connection on the bundle. The Wilson line is then defined as the integral

$$W_\gamma(p, q) := Tr\left[\mathcal{P} \exp\left(\int_\gamma A_i dx^i\right)\right]. \tag{3.105}$$

$\mathcal{P}$ is the path ordering operator. This quantity is not a gauge-invariant object, unless we define a closed curve. The Wilson loop is precisely such a gauge-invariant observable. Given an irreducible representation ρ of the gauge group, we have

$$W_\rho[\gamma, p] := Tr_\rho\left[\mathcal{P} \exp\left(\oint_\gamma A_i dx^i\right)\right]. \tag{3.106}$$

We can now consider $a = 1, ..., s$ such loops γ_a and define the following functional integral

$$\int \mathcal{D}A_i \; e^{ikT} \prod_{a=1}^{s} W_{\rho_a}[\gamma_a], \tag{3.107}$$

where k is the Chern-Simons level. This will be quantized by requiring the functional integral to be gauge-invariant. The functional measure $\mathcal{D}A_i$ refers to a functional integral over all gauge orbits, that is, integration over all equivalence classes of connections modulo gauge transformations. One might expect to see similarities with Feynman's path integration upon replacing the unimodular time with the classical action, but we are not considering such an integration. We shall then define some sort of expectation value (without reference to correlators in a vacuum state) of these Wilson loops as

$$\left\{ \prod_{a=1}^{s} W_{\rho_a}[\gamma_a] \right\} := \frac{\int \mathcal{D}A_i \; e^{ikT} \prod_{a=1}^{s} W_{\rho_a}[\gamma_a]}{\int \mathcal{D}A_i \; e^{ikT}}. \tag{3.108}$$

Notice the similarities and contrasts with the interpretations. Witten considers the full expectation value of the Wilson loops with the action given by the Chern-Simons theory in a $2+1$-dimensional manifold, whereas the present paper deals with a fully spacial 3-manifold on which the Chern-Simons action is replaced with the unimodular time. The striking difference is the presence of the functional integral on the right-hand side of Eq.(3.108) which has again similarities with the standard Feynman path integration, weighted with the Wilson loops. We thus have to first give a physical meaning to the left-hand side, i.e. the expectation value. Thus we are not performing a quantum field theoretic computation, but rather a mathematical trick to compute these knots associated to the unimodular time. A further advantage of the given integral is that there is no need to perform a Wick rotation since everything is purely spacial. This has fundamental differences with the standard path integral approach.

However, computation of the integral is identical to the result obtained in [68]. For the abelian case with $\Sigma = S^3$, one has the very simple result

$$\left\{ \prod_{a=1}^{s} W_{\rho_a}[\gamma_a] \right\} \sim e^{i \sum_{a,b} n_a n_b Lk(\gamma_a, \gamma_b)}, \tag{3.109}$$

where $Lk(\gamma_a, \gamma_b)$ is the Gauss linking number of two links γ_a and γ_b, with $a \neq b$; and n_a, n_b are the labels of the $U(1)$-irreducible representations of each Wilson loop. The case of the loops being the same corresponds to computing knots, which has also been explored by Witten and gives knot invariants and their expectation values. We thus see that there is a strong link between the topological properties of the unimodular time with invariant quantities appearing within knot theory. A further remark concerns about the hypersurface three-manifold on which the functional integration is taking place: each of the polynomials are obtained with regard to a constant-slice, and thus for later times should evolve, otherwise the unimodular condition is does not hold. Thus we require to perform such integrals through all times. This is left to be fully explored for future work.

In the next chapter, we would like to change gears and look back at how unimodular time can be appropriately quantized within Page-Wootters formalism. This will not be fully done, and only be pointed out for future work. We will also be extending the formalism to include relativistic scenarios, which are of relevance for unimodular gravity.

Chapter 4

Page-Wootters Framework

Usually, external structures such as reference frames are used in order to explain matter and motion, and itself being a physical system must also be subject to dynamics. In fact, general relativity takes the concept of clocks and rods forming a reference frame into a single entity, spacetime, which is dynamical and obeys Einstein's field equations. Therefore, it must also be quantized which is a very serious issue as reference frames are not usually quantized within QFT. This issue must be addressed and various such attempts are made within the literature such as [64, 57, 17, 4, 3, 31, 65, 59, 46, 58, 30, 56].

In a relativistic scenario, physical quantities must be Lorentz-invariant. This being fulfilled within the Lagrangian formalism, it is most definitely not within the canonical Hamiltonian formalism, where time is being singled out from the rest of the physical system. This, in turn, creates a series of major problems like the problem of time within quantum gravity [64, 31, 57, 30, 54, 69, 29]. Within gravity, we have shown in the past two chapters that it can be avoided with unimodular time. However, a more general framework must be introduced within field theory in order to avoid such problems for all sorts of theories. This can be achieved by instead using a covariant Hamiltonian formulation of classical field theory, namely the De Donder-Weyl formalism (DW) where time is not singled out and is Lorentz-covariant in all physical quantities. There is a wide range of literature on the DW-formalism mostly due to Kanatchikov and others [16, 40, 41, 39, 66, 62, 26], but is not used quite often within QFT.

As stated throughout this dissertation, time plays a major role in both classical and quantum theories. In the Hamiltonian formalism of classical mechanics, time enters as an evolution parameter of the phase space. In quantum mechanics, time is also considered as a parameter of evolution: in the Schrodinger picture it describes how the Hilbert space evolves in time, while in the Heisenberg picture, it describes how the space of linear operators acting on the Hilbert space evolves in time. This creates a series of issues upon considering a relativistic setting: rods and clocks must be on equal footing. This suggests that time must be resigned from being a parameter but rather be considered as a coordinate by itself. In the Hamiltonian formulation, this means to extend the configuration space by taking time and the generalized coordinates as new independent variables, and for every independent variable of the configuration space, there is an associated canonical conjugate momenta; which in turn extends the phase space as well. With the introduction of such new independent variables on the phase space, Dirac [19, 18] showed that the Hamiltonian function of such a dynamical system must be identically zero due to the presence of constraints. Indeed, we will show that the relativistic free particle propagating in a flat background spacetime is a constrained dynamical system; and extend the formalism to quantum mechanics via constraint operators acting on a kinematical Hilbert space [64, 57, 17, 54, 30]. This will naturally lead to factorizing the Hilbert space into a system and clocks/rods factors and in the continuum limit to quantum fields and perhaps quantum gravity. Note that by Hilbert space factors we mean a quantum system where we quantize the physical dynamical system and the reference frame consisting of clocks and rods. In fact, the problem of time mentioned above comes from the fact that because there is no manifest background dependence, physical states do not depend on time, i.e. the dynamical system is frozen in time. The apparent time-evolution of physical states is merely effective and depends on the internal observers, namely with respect to an idealized reference frame, which is exactly what happens in almost all theories of physics except GR: the biggest lesson is that reference frames are merely effective and internal observer-dependent notions. A reference frame must be relational, relative to which the physical system must evolve, for which the true picture is clocks and rods-neutral rather than timeless.

We will focus on two main aspects of such a relational construction: we will look at the dynamics of relativistic quantum point particles and then extend it to the dynamics of relativistic quantum fields. In both cases, the only thing that changes mainly is the dynamical system Hilbert space factor of the kinematical Hilbert space, which we shall discuss in great detail. The construction is a relativistic extension of the *Page-Wootters construction*, also called the *conditional probability interpretation* of quantum mechanics [54, 69].

4.1 Constrained relational classical dynamics

In this section, we will study closely classical dynamics of relativistic point-particles and classical fields within the context of re-parametrization invariance. For the latter, it will turn out that the canonical Hamiltonian formulation is not good enough to preserve Lorentz-covariance and hence a covariant Hamiltonian formulation must be adopted. This will turn out to be the DW-formalism.

4.1.1 Relativistic free point-particle and re-parametrization invariance

We will start by considering the dynamics of a relativistic free point-particle propagating in a flat background spacetime. We consider the metric signature $\eta_{ij} := \mathrm{diag}(+,-,-,-)$, in order to be consistent with most QFT literature. The action reads:

$$S[X] := -m \int_{\mathbb{R}} dT \sqrt{1 - \left(\frac{dX}{dT}\right)^2}. \tag{4.1}$$

This action is clearly treating time as a parameter, which is not relativistically ideal. One can rewrite this upon introducing a parameter along the curve $x^i = x^i(\lambda)$ which usually is taken to be an affine parameter, where $x^i = (T, X)$. Then the action takes a very interesting form:

$$\begin{aligned} S[x^i] &= -m \int_{\mathbb{R}} d\lambda \dot{T} \sqrt{1 - \left(\frac{\dot{X}}{\dot{T}}\right)^2} \\ &= -m \int_{\mathbb{R}} d\lambda \sqrt{\dot{T}^2 - \dot{X}^2}. \end{aligned} \tag{4.2}$$

The dot-operation refers here to the derivative with respect to the affine parameter, i.e. $\dot{f} := \frac{df}{d\lambda}$. Using the Minkowski metric tensor as our line-element definition:

$$\begin{aligned} \eta :&= \eta_{ij} dx^i \otimes dx^j \\ &= \frac{1}{2}\left(\eta_{ij} dx^i \otimes dx^j + \eta_{ji} dx^j \otimes dx^i\right) \\ &= \frac{1}{2}\left(\eta_{ij} dx^i \otimes dx^j + \eta_{ii} dx^j \otimes dx^i\right) \\ &= \eta_{ij} dx^i dx^j \equiv ds^2, \end{aligned}$$

one can write

$$ds = \sqrt{\eta_{ij} \dot{x}^i \dot{x}^j} d\lambda = d\lambda \sqrt{\dot{T}^2 - \dot{X}^2}.$$

Hence the action simply reads:

$$S[x^i] = -m \int_{\mathcal{M}} ds \equiv -m \int_{\mathbb{R}} d\lambda \sqrt{\eta_{ij} \dot{x}^i \dot{x}^j}. \tag{4.3}$$

It can be seen that this action is not only Lorentz invariant but also space and time are on equal footing. The dynamics of the particle are given by the time evolution of the configuration space with respect to the affine parameter λ. We will call this the flow of the configuration space. Notice that the action is also re-parametrization invariant, i.e. under the change of parameter $\lambda \to f(\lambda) = \tilde{\lambda}$, the action remains invariant. This indicates that by putting clocks and rods on equal footing, we have an extended configuration space $\mathcal{C}_{\text{ext}}$ with independent variables $(T, X) \equiv x^i \in \mathcal{C}_{\text{ext}}$. The time-evolution of the extended configuration space is given through the affine parameter we introduced, which we call the *flow* of the extended configuration space. It is clear that the system is frozen with respect to the clock variable T but is unfrozen with respect to the affine parameter λ. However, the affine parameter being arbitrarily chosen, there is no real physical meaning that has to be associated to it. This is the reason it is an arbitrary evolution parameter of the extended configuration space.

In going from the Lagrangian formulation to the Hamiltonian formulation, one has to find the associated canonical momentums. For our dynamical system, we have :

$$x^i \longrightarrow p_i := \frac{\partial L}{\partial \dot{x}^i} = -\frac{m\dot{x}_i}{\sqrt{\eta_{ab}\dot{x}^a\dot{x}^b}} = -m\dot{x}_i, \tag{4.4}$$

where we used the fact that because λ is an affine parameter, $\sqrt{\eta_{ab}\dot{x}^a\dot{x}^b} = 1$. In fact we have a naturally extended phase space $\mathcal{P}_{\text{ext}}$ as the variables are $(x^i, p_i) \equiv (T, X, p_T, p_X) \in \mathcal{P}_{\text{ext}}$. We thus have not only a canonical momentum associated with the rod/position variable X but also a canonical momentum associated with the clock/time variable T. In fact Eq.(4.4) can be written as:

$$\begin{aligned} p_0 &= p_T = -m\dot{T}, \\ p_\alpha &= -p_X = m\dot{X}. \end{aligned}$$

This clearly shows that there is a clock canonical momentum proportional to the rate of change of the clock with respect to the flow of the extended configuration space. In this Hamiltonian picture, one can perform a Legendre transform of the Lagrangian function given in Eq.(4.3) to obtain the Hamiltonian, namely

$$\begin{aligned} H(x^i, p_i) :&= p_i\dot{x}^i - L(x^i, \dot{x}^i) \\ &= -\frac{1}{m}p_ip^i + m\sqrt{\eta_{ab}\dot{x}^a\dot{x}^b} \\ &= -\frac{1}{m}\left(p_ip^i - m^2\right). \end{aligned} \tag{4.5}$$

Also,

$$\begin{aligned} H(x^i, p_i) :&= p_i\dot{x}^i - L(x^i, \dot{x}^i) \\ &= -\frac{m\dot{x}_i\dot{x}^i}{\sqrt{\eta_{ab}\dot{x}^a\dot{x}^b}} + m\sqrt{\eta_{ab}\dot{x}^a\dot{x}^b} = 0. \end{aligned} \tag{4.6}$$

Combining Eq.(4.5) and Eq.(4.6) one end-ups with the very nice formula

$$p_ip^i - m^2 = 0, \tag{4.7}$$

which is the *mass-shell hyperboloid condition*. It should also be noted that the Hamiltonian vanishes. This suggests that we have indeed a constrained dynamical system, with the constraint being the mass-shell hyperboloid condition, i.e.

$$C(x^i, p_i) := p_ip^i - m^2, \tag{4.8}$$

such that the constraint vanishes, namely $C = 0$. This constraint is usually defined on a hyper-surface of the extended phase space. We can thus introduce this constraint via a Lagrange multiplier called the lapse function N which for our case is given by $N \equiv -1/m$ into Eq.(4.3) upon Legendre transforming the Lagrangian function:

$$S[x^i, p_i, N] := \int_{\mathbb{R}} d\lambda \left(p_i\dot{x}^i + NC\right). \tag{4.9}$$

Varying the action with respect to x^i leads to the conservation of the four-canonical momentum. Varying the action with respect to the lapse function N leads to the constraint given in Eq.(4.8) which is the mass-shell hyperboloid condition. One can then define the following extended Poisson brackets:

$$\dot{x}^i = \{x^i, -NC\} = -2Np^i \Longleftrightarrow p^i = -\frac{\dot{x}^i}{2N}, \tag{4.10}$$

$$\dot{p}_i = \{p_i, -NC\} = 0 \Longleftrightarrow p_i = c \equiv -m\dot{x}_i, \tag{4.11}$$

$$\dot{N} = \{N, -NC\} = 0 \Longleftrightarrow N = d \equiv -\frac{1}{2m}. \tag{4.12}$$

Here, c and d are constants, as required by the vanishing time derivatives. The action given by Eq.(4.9) may thus be rewritten as follows:

$$\begin{aligned} S &= \int_{\mathbb{R}} d\lambda \left(\frac{1}{2} p_i \dot{x}^i + \frac{1}{2} p_i \dot{x}^i + N p_i p^i - N m^2 \right) \\ &= \int_{\mathbb{R}} d\lambda \left(-\frac{1}{4N} \dot{x}_i \dot{x}^i - N m^2 \right) \\ &= -\frac{1}{2} \int_{\mathbb{R}} d\lambda \left(\frac{1}{2N} \dot{x}_i \dot{x}^i + 2N m^2 \right). \end{aligned} \tag{4.13}$$

This action is a Polyakov-type action for the relativistic free-particle which clearly works for massless particles and is classically equivalent to the original action given by Eq.(4.3).

To sum up, we have seen that one can introduce a parameter λ to a curve and this results in de-parametrizing the clock time T and extending our phase space. As a consequence, one gets a constrained dynamical system with the constraint given by the mass-shell hyperboloid condition, i.e. particle masses are bounded to the hyperboloid. Notice that dynamics of the re-parametrization invariant formulation of relativistic free point-particle seems to suggest that there is no time-evolution on the extended phase space, but appears effectively on each constant hyper-surface of the extended phase space. This is exactly what is the essence of the constraint formulation and what we hope to understand better within the quantum formulation. In the next section, we will give the postulates for a constrained relational quantum dynamical theory and give physical meaning to all this within the framework of quantum mechanics.

4.2 Foundations of the CQR-dynamics

In this section, we lay the foundations for the constrained relational quantum dynamics by closely following and extending the work done in [30].

4.2.1 The extended Hilbert space

The very first point to have in mind is that we require a generally covariant description for our dynamical theory to have. This can be achieved if and only if time and space variables are on equal footing and the theory in consideration is generally diffeomorphism-invariant. For instance, in the special theory of relativity (SR), there exists the concept of a reference frame consisting of clocks and rods as measuring devices such that the dynamics are given in a relational manner, i.e. with respect to change of reference frames and how they relate to one another. Such reference frames are usually Lorentz or Poincaré-invariant, which is the diffeomorphism-invariance within SR. It is clear that given a mechanical system, clocks and rods do not interact with each other trivially. In our daily lives, we do not consider the composite system necessarily and contain ourselves with clocks as time-measuring devices in order to describe the evolution of dynamical systems. However, as can be seen in relativistic and quantum mechanical situations, there has to be a precise meaning to what a clock is. In fact, it is better to consider clocks and rods as non-interacting composite systems. These frames are not only defined locally but also globally on the entire spacetime manifold. In a quantum theory, this must translate into a *quantum reference frame* which we will represent mathematically by the *extended Hilbert space* defined by:

$$\mathcal{H}_{\text{ext}} := \mathcal{H}_{\text{T}} \otimes \mathcal{H}_{\text{X}}, \tag{4.14}$$

where $\mathcal{H}_{\text{T}} \cong \mathcal{L}(\mathbb{R}, d\mu_{\text{T}})$ is the *clock Hilbert space* and $\mathcal{H}_{\text{X}} \cong \mathcal{L}(\mathbb{R}^3, d\mu_{\text{X}})$ is the *rods Hilbert space*. Notice that this type of factorisation leads to the notion of composite systems, where clocks and rods are considered to form a composite system and these are locally and globally defined within the framework of SR of course, whereas in GR only locally may this quantum reference be defined trivially. As transformations between different Hilbert spaces are carried through unitary operators, quantum reference frames are related by unitary transformations of the form

$$\widehat{U}_{\text{ext}} : \mathcal{H}_{\text{ext}} \longrightarrow \mathcal{H}'_{\text{ext}}, \tag{4.15}$$

such that the inner-product on $\mathcal{H}_{\text{ext}}$ is the same as in $\mathcal{H}'_{\text{ext}}$, i.e. the inner-product is preserved. To be more precise, one does make transformations between clocks and rods such that there exists unitary operators

$$\widehat{U}_{\mathrm{T}} : \mathcal{H}_{\mathrm{T}} \longrightarrow \mathcal{H}'_{\mathrm{T}} \quad \text{and} \quad \widehat{U}_{\mathrm{X}} : \mathcal{H}_{\mathrm{X}} \longrightarrow \mathcal{H}'_{\mathrm{X}}, \tag{4.16}$$

where $\hat{U}_{\mathrm{T}}$ is the unitary transformation on the clock Hilbert space and $\hat{U}_{\mathrm{X}}$ is the unitary transformation on the rods Hilbert space. Then one can write the extended unitary transformation operator as

$$\widehat{U}_{\text{ext}} := \widehat{U}_{\mathrm{T}} \otimes \widehat{U}_{\mathrm{X}}. \tag{4.17}$$

One can think of the unitary operator $\widehat{U}_{\text{ext}}$ as different observers relating one quantum reference frame to another.

The state of a quantum reference frame is mathematically given by state vectors $|\psi_{\text{ext}}\rangle \in \mathcal{H}_{\text{ext}}$ such that for all $|\psi_{\mathrm{T}}\rangle \in \mathcal{H}_{\mathrm{T}}$ and $|\psi_{\mathrm{X}}\rangle \in \mathcal{H}_{\mathrm{X}}$

$$|\psi_{\text{ext}}\rangle := |\psi_{\mathrm{T}}\rangle \otimes |\psi_{\mathrm{X}}\rangle \equiv |\psi_{\mathrm{T}}, \psi_{\mathrm{X}}\rangle \,. \tag{4.18}$$

Using Eq.(4.17) and Eq.(4.18), one gets

$$\widehat{U}_{\text{ext}} |\psi_{\text{ext}}\rangle = |\psi'_{\text{ext}}\rangle \,, \tag{4.19}$$

or

$$\begin{aligned} \widehat{U}_{\text{ext}} |\psi_{\mathrm{T}}, \psi_{\mathrm{X}}\rangle &= \left(\widehat{U}_{\mathrm{T}} \otimes \widehat{U}_{\mathrm{X}}\right) |\psi_{\mathrm{T}}, \psi_{\mathrm{X}}\rangle \\ &= |\psi'_{\mathrm{T}}, \psi'_{\mathrm{X}}\rangle \equiv |\psi'_{\text{ext}}\rangle \,. \end{aligned} \tag{4.20}$$

Notice that a change of reference frame in SR must be under Lorentz or Poincaré transformations. The unitary transformation $\widehat{U}_{\text{ext}}$ is thus a mapping from the Poincaré group to the Lie group of linear isomorphisms on $\mathcal{H}_{\text{ext}}$. In the context of GR, the transformation must be a diffeomorphic transformation i.e. the unitary transformation is a mapping from the diffeomorphism group to the Lie group of linear isomorphisms on $\mathcal{H}_{\text{ext}}$. The class of different diffeomorphism groups being wide, studying the system locally must always lead to considering the more restrictive class, the Poincaré group. Also, diffeomorphisms within GR are local gauge transformations, which means that they are not very well-suitable for defining a projective unitary representation. That is why, we will only consider projective unitary transformations of the Poincaré group locally if we deal with general relativistic situations.

With such a quantum reference frame defined locally, one can start thinking about the associated quantum clock and rod observables. First of all, they must be linear operators acting on the extended Hilbert space, i.e. there must exist *time* and *position operators* acting on the extended Hilbert space. However, it is known that according to Pauli's argument [55], there cannot exist a time operator canonical to the Hamiltonian of a dynamical system as the latter is an operator bounded from below. One can bypass the problem within the Page-Wootters formulation [54, 69] where instead of the self-adjointness condition for operators, a positive operator-valued measure (POVM) is introduced instead. It is well known that quantization is the procedure of mapping classical observables into quantum observables, which are represented mathematically by linear operators acting on Hilbert spaces. We will denote such a map by $\gamma : \mathcal{P}_{\text{ext}} \to \mathcal{L}(\mathcal{H}_{\text{ext}})$, where $\mathcal{H}_{\text{ext}}$ is the Hilbert space over the complex numbers $\mathbb{C}$ and $\mathcal{L}(\mathcal{H}_{\text{ext}})$ is the space of linear operators acting on $\mathcal{H}_{\text{ext}}$. Thus we have the very natural mappings:

$$\begin{aligned} \gamma &: x^i \mapsto \widehat{x}^i, \\ \gamma &: p_i \mapsto \widehat{p}_i. \end{aligned}$$

Then, according to Stone's theorem, the canonically defined four-momentum operator generates spacetime translations according to:

$$\widehat{U}(a) := e^{ia^i \widehat{p}_i}, \tag{4.21}$$

where $\widehat{p}_i$ are now necessarily self-adjoint and $\forall a \in \mathbb{P}$, the Poincaré group–to be more precise we only consider the subgroup of spacetime translations only, i.e. we refer to the spacetime translations group as the Poincaré. One can define the conjugate operators by introducing the POVM $\widehat{E}_x$ such that the *covariance* condition is satisfied, namely

$$\widehat{E}_x(A+a) = \widehat{U}(a)\widehat{E}_x(A)\widehat{U}^\dagger(a), \tag{4.22}$$

where $A \in \mathrm{Bor}(\mathbb{P})$ is the Borel σ-algebra of the Poincaré group. The POVM $\widehat{E}_x$ is referred to be the *four-position observable* and is self-adjoint by construction such that for *clock/rods states* $|x\rangle := |T, X\rangle \in \mathcal{H}_{\text{ext}}$ the POVM is

$$\widehat{E}_x(d^4x) = \mu\, d^4x\, |T, X\rangle \langle T, X| \,. \tag{4.23}$$

Hence, one has

$$\widehat{E}_x(A) = \int_A \widehat{E}_x(d^4x). \tag{4.24}$$

The resolution of the identity is then nothing but the normalisation condition for the POVM axiom, namely

$$\widehat{E}_x(\mathbb{P}) = \int_{\mathbb{P}} \widehat{E}_x(d^4x) \equiv \widehat{\mathrm{id}}_{\mathcal{H}_{\text{ext}}}.$$

The covariance condition in Eq.(4.22) then implies that $|a\rangle = \widehat{U}(a-x)\,|x\rangle$ and this results in defining the *four-position operator* as the first moment of the moment operator of the four-position observable:

$$\widehat{x}^i := \mu \int_{\mathbb{P}} d^4x\, x^i\, |T, X\rangle \langle T, X| \,. \tag{4.25}$$

This suggests that the four-position operator is necessarily symmetric but might not be self-adjoint. The important lesson is that the associated quantum observable is not $\widehat{x}^i$ but rather $\widehat{E}_x$. This together with Stone-von Neumann theorem leads to the canonical commutation relations (CCR):

$$[\widehat{x}^i, \widehat{p}_j] := i\delta^i_j\, \widehat{\mathrm{id}}_{\mathcal{H}_{\text{ext}}}. \tag{4.26}$$

It can be seen that from Heisenberg's uncertainty principle -verified experimentally on numerous occasions- there exists an uncertainty between position and momentum such that

$$\Delta X \Delta p_X \geq \frac{1}{2}. \tag{4.27}$$

This translates mathematically that the associated linear operators $\widehat{X}$ and $\widehat{p}_X$ are incompatible quantum observables, i.e.

$$[\widehat{x}^\alpha, \widehat{p}_\beta] = i\delta^\alpha_\beta\, \widehat{\mathrm{id}}_{\mathcal{H}_X}. \tag{4.28}$$

This also suggests that position and momentum are related upon Fourier transform, as expected. A controversial topic is the time-energy uncertainty relation which states that

$$\Delta T \Delta E \geq \frac{1}{2}. \tag{4.29}$$

This in principle must lead to a CCR of the form

$$[\widehat{T}, \widehat{H}] = i\, \widehat{\mathrm{id}}_{\mathcal{H}_{\mathrm{T}}}. \tag{4.30}$$

However, according to Pauli [55] there cannot exist such a time operator $\widehat{T}$ as the Hamiltonian in standard quantum mechanics is bounded from below, whereas time is unbounded which suggests that they cannot act on the same Hilbert space. Also, it would violate the Stone-von Neumann theorem which is an unacceptable result. Therefore one assumes in standard quantum mechanics that time must remain as a parameter of evolution satisfying either Schrodinger's or Heisenberg's pictures. However, as argued above, SR requires putting clocks and rods on the same footing and to do this we used an extended phase space such that we end up with a constrained dynamical system. If a mapping γ exists, then this should suggest that there is also an extended Hilbert space associated with the extended phase space, which was described above. In order to determine this Hilbert space, we have to understand clocks and rods much closer. In order to achieve this, we will consider the Hilbert space defined in Eq.(4.15) and define the four-position linear operators $\hat{x}^i$ with components

$$\widehat{x}^0 := \widehat{T} \otimes \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{X}}, \tag{4.31}$$

$$\widehat{x}^\alpha := \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{T}} \otimes \widehat{X}^\alpha. \tag{4.32}$$

The associated canonical four-momenta linear operator $\hat{p}_i$ has components

$$\widehat{p}_0 := \widehat{p}_T \otimes \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{X}}, \tag{4.33}$$

$$\widehat{p}_\alpha := \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{T}} \otimes \widehat{p}_{X^\alpha}. \tag{4.34}$$

The CCR is then defined on the extended phase space according to Eq.(4.26) with components given by:

$$\begin{aligned}[\widehat{x}^0, \widehat{p}_0] &= [\widehat{T} \otimes \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{X}}, \widehat{p}_T \otimes \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{X}}] \\ &= [\widehat{T}, \widehat{p}_T] \otimes \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{X}} = i\ \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{T}} \otimes \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{X}} \\ &= i\ \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{ext}},\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}[\widehat{x}^\alpha, \widehat{p}_\beta] &= [\widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{T}} \otimes \widehat{X}^\alpha, \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{T}} \otimes \widehat{p}_{X^\alpha}] \\ &= \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{T}} \otimes [\widehat{X}^\alpha, \widehat{p}_{X^\beta}] = i\delta^\alpha_\beta\ \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{T}} \otimes \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{X}} \\ &= i\delta^\alpha_\beta\ \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{ext}}.\end{aligned} \tag{4.36}$$

Notice that we do not require any Hamiltonian operator for our system, yet. We will see that the Hamiltonian enters in a much different setting with a beautiful meaning. The only message we have to get from here is that the uncertainty relations read:

$$\begin{aligned}\Delta T \Delta p_T &\geq \frac{1}{2}, \\ \Delta X \Delta p_X &\geq \frac{1}{2}.\end{aligned} \tag{4.37}$$

We can interpret it as saying that the less uncertainty we read in a clock's measurement, the more uncertainty we will have to read in the clock's momentum. We thus do not have any time-energy uncertainty relation, but instead, we end-up with a "clock-clock momentum" uncertainty relation which has more meaning. However, to understand the role of energy, i.e. the Hamiltonian, we have to step back and think about what we previously found.

Before we go any further, we have to specify an appropriate inner product on the extended Hilbert space. Consider the following inner product

$$\begin{aligned}\langle \psi_\mathrm{ext} | \phi_\mathrm{ext} \rangle &:= \langle \psi_\mathrm{ext} |\ \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{ext}} \cdot \widehat{\mathrm{id}}_{\mathcal{H}_\mathrm{ext}}\ | \phi_\mathrm{ext} \rangle \\ &= \int \psi^*_\mathrm{ext}\,(T, X)\,\phi_\mathrm{ext}\,(T, X)\, d^4x,\end{aligned} \tag{4.38}$$

where we used the fact that $\langle T, X|\psi_{\text{ext}}\rangle \equiv \psi_{\text{ext}}(T, X) \in \mathbb{C}$. However, another type of inner product might also be specified which is conditional. For instance, one can define the object

$$\left(\langle T| \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{X}}}\right) |\psi_{\text{ext}}\rangle =: |\psi_X(T)\rangle \in \mathcal{H}_{\text{X}}. \tag{4.39}$$

We thus have an expansion of the sort

$$|\psi_{\text{ext}}\rangle = \int dT \, |\psi_X(T)\rangle \, |T\rangle \, .$$

In that case, an appropriate inner product on the extended Hilbert space is given by

$$\begin{aligned} \langle \psi_{\text{ext}}|\phi_{\text{ext}}\rangle : &= \langle \psi_{\text{ext}}| \left(|T\rangle \langle T| \otimes \widehat{\text{id}}_{\mathcal{H}_X}\right) |\phi_{\text{ext}}\rangle \\ &= \langle \psi_X(T)|\phi_X(T)\rangle \, , \end{aligned} \tag{4.40}$$

which ensures that the probability density is conserved in time. Nevertheless, this naive approach is not a correct description of how probabilities emerge. This is what we will do next.

4.2.2 The system factor and the physical Hilbert space

As mentioned above, there exists a quantum reference frame mathematically represented by the extended Hilbert space $\mathcal{H}_{\text{ext}}$. However, a reference frame is usually not enough, as there also exists a *system* on which the reference frame acts and is used to describe dynamics. This is exactly what we need in order to define a Hamiltonian for the dynamical system. In terms of quantum theory, this will be given by the *system Hilbert space* $\mathcal{H}_{\text{sys}}$. This system and the reference must interact in a dynamical fashion, i.e. we have to consider instead the composite system

$$\mathcal{H}_{\text{kin}} \cong \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{ext}}. \tag{4.41}$$

We therefore not only have non-interacting correlations between clocks and rods within the reference frame, but also interacting correlations between the system and the reference frame. The right-hand side of Eq.(4.41) is the *kinematical Hilbert space* and is shown to be isomorphic to the composite quantum dynamical system. Specific choices of a system Hilbert space would lead to specific dynamical systems. States $|\Psi_{\text{kin}}\rangle \in \mathcal{H}_{\text{kin}}$ are called *kinematical states*, on which appropriately chosen constraint operators $\widehat{C}$ must act on. For states to be physical, one has to further satisfy the condition that constraint operators must annihilate kinematical states. Such states $|\Psi_{\text{phy}}\rangle \in \mathcal{H}_{\text{phy}}$ are called *physical states* and the kinematical Hilbert space is called a *physical Hilbert space*. With such a structure, one has to further define an inner product on the physical Hilbert space. We will define such an inner product soon, but before that, let us examine what happens to physical states upon a change of reference frame. The first thing to do is to extend the unitary transformation operator given in Eq.(4.16) and Eq.(4.20) to the physical Hilbert space, i.e. we consider the unitary operator

$$\widehat{U}_{\text{phy}} := \widehat{\text{id}}_{\mathcal{H}_{\text{sys}}} \otimes \widehat{U}_{\text{ext}}, \tag{4.42}$$

such that its action on physical states $|\Psi_{\text{phy}}\rangle \equiv |\psi_{\text{sys}}, \psi_{\text{ext}}\rangle$ is given by

$$\begin{aligned} \widehat{U}_{\text{phy}} |\Psi_{\text{phy}}\rangle &= \left(\widehat{\text{id}}_{\mathcal{H}_{\text{sys}}} \otimes \widehat{U}_{\text{ext}}\right) |\psi_{\text{sys}}, \psi_{\text{ext}}\rangle \\ &= |\psi_{\text{sys}}, \psi'_{\text{ext}}\rangle \equiv |\Psi'_{\text{phy}}\rangle \, . \end{aligned} \tag{4.43}$$

Thus the system's state remains unchanged, whereas the extended states are subject to transformation. The composite product is then changed according to Eq.(4.43). However, instead of a reference frame transformation, one could also have unitary transformations within the system itself. A combination of both would lead to the fully unitary transformation

$$\widehat{U}_{\text{phy}} |\Psi_{\text{phy}}\rangle = |\Psi'_{\text{phy}}\rangle \equiv |\psi'_{\text{sys}}, \psi'_{\text{ext}}\rangle \, , \tag{4.44}$$

where we have the unitary operator

$$\widehat{U}_{\text{phy}} := \widehat{U}_{\text{sys}} \otimes \widehat{U}_{\text{ext}} \equiv \widehat{U}_{\text{sys}} \otimes \widehat{U}_{\text{T}} \otimes \widehat{U}_{\text{X}}. \tag{4.45}$$

Both unitary operators associated with $\widehat{U}_{\text{sys}}$ and $\widehat{U}_{\text{ext}}$ are a result of two different Poincaré (or translations in four dimension for our case) groups, one which acts on $\mathcal{H}_{\text{ext}}$ and the other on $\mathcal{H}_{\text{sys}}$. These in turn will result in two distinct Casimir invariants, namely $\widehat{P}^2$ and $\widehat{p}^2$, where the former gives mass to one particle state and the latter gives the wave operator ∂^2. Let us now examine the individual factors and states of each factor. Let us start with the most important one, the clock Hilbert space $\mathcal{H}_{\text{T}}$: we are tempted to define time as a measurement of the time operator $\widehat{T}$ on $\mathcal{H}_{\text{T}}$ which is taken to be canonically conjugate to the canonical clock-momentum $\widehat{p}_T$ satisfying the CCR $[\widehat{T}, \widehat{p}_T] = i\ \widehat{\text{id}}_{\mathcal{H}_{\text{T}}}$. For such a CCR, states of the clock indicating different times correspond to eigenstates $|T\rangle$ of the time operator with time indicated by the clock corresponding to the eigenvalues $T \in \mathbb{R}$. In turn, one can translate infinitesimally the time operator via

$$e^{iS\widehat{p}_T}\ \widehat{T}\ e^{-iS\widehat{p}_T} := \widehat{T} + S\ \widehat{\text{id}}_{\mathcal{H}_{\text{T}}}, \tag{4.46}$$

where we used the Baker-Campbell-Hausdorff formula. This then makes it possible to define the resolution of the identity

$$\widehat{\text{id}}_{\mathcal{H}_{\text{T}}} := \mu \int dT\, |T\rangle \langle T|\,, \tag{4.47}$$

such that the spectral theorem for the time operator can be stated simply as a POVM, namely

$$\widehat{T} := \mu \int dT\ T\, |T\rangle \langle T|\,. \tag{4.48}$$

This is analogous to the four-position operator defined in Eq.(4.25). We thus can rewrite the infinitesimal translation in time as

$$e^{i\widehat{p}_T S}\, |T\rangle = |T + S\rangle\,. \tag{4.49}$$

Thus the canonical-clock operator generates infinitesimal translations. Such time eigenstates will be referred to as the *clock states*. We will usually take the clock states to be eigenstates of the time operator such that they satisfy the orthogonality condition $\langle T'|T\rangle := \delta(T' - T)$. This will ensure that the clock states are perfectly distinguishable and that the clock does not run backwards in time, i.e. $\langle T'|\, e^{i\widehat{p}_T S}\, |T\rangle = 0, \forall S > 0$ if $T > T'$.

A similar treatment for the rods/spatial Hilbert space factor $\mathcal{H}_{\text{X}}$ can be constructed: we are tempted to define position as a measurement of the position operator $\widehat{X}$ on $\mathcal{H}_{\text{X}}$ which is taken to be canonically conjugate to the canonical position-momentum $\widehat{p}_X$ satisfying the CCR $[\widehat{X}, \widehat{p}_X] = -i\ \widehat{\text{id}}_{\mathcal{H}_{\text{X}}}$–notice the minus sign, which is due to the fact that we have a spacial covariant indexed canonical three momentum. For such a CCR, states of the rods indicating different position correspond to eigenstates $|X\rangle$ of the position operator with position indicated by the rods corresponding to the eigenvalues X. In turn, one can translate infinitesimally the position operator via

$$e^{-iA\widehat{p}_X}\widehat{X}e^{iA\widehat{p}_X} := \widehat{X} + A\ \widehat{\text{id}}_{\mathcal{H}_{\text{X}}}, \tag{4.50}$$

where we used the Baker-Campbell-Hausdorff formula. This then makes it possible to define the resolution of the identity

$$\widehat{\text{id}}_{\mathcal{H}_{\text{X}}} := \int d^3X\, |X\rangle \langle X|\,, \tag{4.51}$$

such that the spectral theorem for the position operator can be stated simply as

$$\widehat{X} := \int d^3X\ X\, |X\rangle \langle X|\,. \tag{4.52}$$

This is again analogous to the four-position operator defined in Eq.(4.25). We thus can rewrite the infinitesimal translation in position as

$$e^{-i\widehat{p}_X A}\,|X\rangle = |X + A\rangle\,. \tag{4.53}$$

Thus the canonical position-momentum operator $\widehat{p}_X$ generates infinitesimal translations. Such position eigenstates will be referred to as the *position states*. We will usually take the position states to be eigenstates of the position operator such that they satisfy the orthogonality condition $\langle X'|X\rangle := \delta(X' - X)$.

We are now able to define an inner product on the physical Hilbert space. First, one has to introduce *conditioned states*: a physical state is said to be conditioned to live on the system Hilbert space if the following mapping exists:

$$\widehat{\mathrm{id}}_{\mathcal{H}_{\text{sys}}} \otimes \langle T, X| : \left|\Psi_{\text{phy}}\right\rangle \mapsto \left|\psi_{\text{sys}}(T, X)\right\rangle. \tag{4.54}$$

This is an extension of the map defined via Eq.(4.39). In turn, this implies that upon using the resolution of the identity on the extended Hilbert space, we get the expansion postulate of physical states:

$$\begin{aligned}
\left|\Psi_{\text{phy}}\right\rangle &= \left(\widehat{\mathrm{id}}_{\mathcal{H}_{\text{sys}}} \otimes \widehat{\mathrm{id}}_{\mathcal{H}_{\text{ext}}}\right)\left|\Psi_{\text{phy}}\right\rangle \\
&= \left(\widehat{\mathrm{id}}_{\mathcal{H}_{\text{sys}}} \otimes \int dT d^3X\, |T, X\rangle\, \langle T, X|\right)\left|\Psi_{\text{phy}}\right\rangle \\
&= \int dT d^3X\, \left|\psi_{\text{sys}}(T, X)\right\rangle |T, X\rangle\,.
\end{aligned} \tag{4.55}$$

One can then define the inner product on $\mathcal{H}_{\text{sys}}$ via

$$\left\langle\Psi_{\text{phy}}\middle|\Phi_{\text{phy}}\right\rangle : = \left\langle\Psi_{\text{phy}}\right|\left(\widehat{\mathrm{id}}_{\mathcal{H}_{\text{sys}}} \otimes |T, X\rangle\, \langle T, X|\right)\left|\Phi_{\text{phy}}\right\rangle \equiv \left\langle\psi_{\text{sys}}(T, X)\middle|\phi_{\text{sys}}(T, X)\right\rangle. \tag{4.56}$$

Thus, we project our physical states onto the system Hilbert space, justifying the term conditional states. This inner product on physical states in turn defines a conditional probability density conservation law, which is required as expected.

As we have the four-position representation, we can also construct the Fourier transform equivalent canonical four-momentum representation, i.e. eigenstates of the canonical four-momentum operator:

$$\langle p_T, p_X| : \widehat{p}_i\, |\psi_{\text{ext}}\rangle \mapsto p_i \psi_{\text{ext}}(p_T, p_X). \tag{4.57}$$

On the system states, we then have

$$\widehat{\mathrm{id}}_{\mathcal{H}_{\text{sys}}} \otimes \langle p_T, p_X| : \left|\Psi_{\text{phy}}\right\rangle \mapsto \left|\psi_{\text{sys}}(p_T, p_X)\right\rangle. \tag{4.58}$$

Thus any physical state might be expanded as

$$\left|\Psi_{\text{phy}}\right\rangle = \int dp_T d^3p_X\, \left|\psi_{\text{sys}}(p_T, p_X)\right\rangle |p_T, p_X\rangle\,. \tag{4.59}$$

The integral is also a sum for the case of a discrete spectrum. We clearly see that the conditioning provided by Eq.(4.54) and Eq.(4.58) are equivalent up to a Fourier transformation. Such maps will be referred to as *reduction/conditioning maps* which condition states to the system Hilbert space factor, namely

$$\left\langle\widehat{\mathcal{R}}_{\text{sys}}\right| := \widehat{\mathrm{id}}_{\mathcal{H}_{\text{sys}}} \otimes \langle T, X| : \mathcal{H}_{\text{phy}} \longrightarrow \mathcal{H}_{\text{sys}}. \tag{4.60}$$

and its Fourier-transformed reduction/conditioning map is

$$\left\langle\mathbf{F}\widehat{\mathcal{R}}_{\text{sys}}\right| := \widehat{\mathrm{id}}_{\mathcal{H}_{\text{sys}}} \otimes \langle p_T, p_X| : \mathcal{H}_{\text{phy}} \longrightarrow \mathcal{H}_{\text{sys}}. \tag{4.61}$$

The only thing left to be defined is the constraint operator itself, namely the mapping which annihilates physical states:

$$\widehat{C} : \left|\Phi_{\text{phy}}\right\rangle \mapsto \widehat{C}\left|\Phi_{\text{phy}}\right\rangle := 0. \tag{4.62}$$

The form of the constraint operator will depend on every dynamical system in question so one must consider various applications, which is what we will do in the next section.

4.3 Applications of the CQR-dynamics

4.3.1 Relativistic free point-particle

For instance, let us focus on relativistic free point particles within the context of SR. According to Eq.(4.8), the propagation of a relativistic free point-particle is a constrained dynamical system for which the constraint is the mass-shell hyperboloid condition. To implement the quantum theory, one has to consider naively the mapping:

$$\gamma : C \mapsto \widehat{C} := \widehat{p}_i\widehat{p}^i - m^2\widehat{\text{id}}_{\mathcal{H}_{\text{ext}}}, \tag{4.63}$$

such that for any state vector $|\psi_{\text{ext}}\rangle \in \mathcal{H}_{\text{ext}}$, the constraint operators action annihilates the state, i.e. $\hat{C}\,|\psi_{\text{ext}}\rangle = 0$. There is a serious problem in doing so as in quantum theory, there is an extra degree of freedom for quantum particles to have, namely spin. In fact, the operator given by $\widehat{p}_i\widehat{p}^i$ is not the Casimir operator associated with the Poincaré group, which gives the mass of all particles in the given representation. However, in order to extract the appropriate Hamiltonian, one has to consider the spin of the particle as a degree of freedom as well. One is then tempted to consider the kinematical Hilbert space

$$\mathcal{H}_{\text{kin}} \cong \mathcal{H}_{\text{s}} \otimes \mathcal{H}_{\text{ext}}, \tag{4.64}$$

where $\mathcal{H}_{\text{s}}$ is the spinorial factor of the Hilbert space corresponding to the system in hand, i.e. relativistic free point-particles with spin; and $\mathcal{H}_{\text{ext}}$ is defined as in Eq.(4.15). We do this in order to extract not only the mass but also the spin of the quantum particle. The Casimir operators are given by the energy-momentum invariant $\widehat{P}_i\widehat{P}^i$ which gives mass and the Pauli-Lubanski invariant $\widehat{W}_i\widehat{W}^i$ which gives spin. In order to get the Hamiltonian of the system, we do not really need the Pauli-Lubanski invariant explicitly, hence we will not use it here. However, the constraint operator defined by Eq.(4.63) must be extended to:

$$\widehat{C} := \widehat{\text{id}}_{\mathcal{H}_{\text{s}}} \otimes \widehat{p}_i\widehat{p}^i + \widehat{P}_i\widehat{P}^i \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}}, \tag{4.65}$$

such that for all $|\Psi\rangle \in \mathcal{H}$ we have

$$\widehat{C}\left|\Psi_{\text{phy}}\right\rangle = 0. \tag{4.66}$$

Clearly, this scalar constraint operator is a strong condition that every relativistic theory of particles must satisfy, namely the mass-shell hyperbolicity. It is therefore essential to find a constraint operator that explicitly displays the energy of the dynamical system and from which the scalar constraint operator defined above may be obtained. We wish to have a constraint operator similar in form to the Page-Wootters constraint given by

$$\widehat{C}_{\text{PW}} := \widehat{\text{id}}_{\mathcal{H}_{\text{s}}} \otimes \widehat{p}_T + \widehat{H}_s \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{T}}},$$

which is non-relativistic of course by definition. We thus wish to construct a constraint operator that transforms correctly under Poincare transformations. With these in hand, let us define the following four-momentum constraint operator as an ansatz:

$$\widehat{C}_i := \widehat{\text{id}}_{\mathcal{H}_{\text{s}}} \otimes \widehat{p}_i + \widehat{P}_i \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}}, \tag{4.67}$$

such that for all physical states $\left|\Psi_{\text{phy}}\right\rangle \in \mathcal{H}_{\text{phy}}$ we have

$$\widehat{C}_i \left|\Psi_{\text{phy}}\right\rangle = 0. \tag{4.68}$$

This is our analogue Wheeler-DeWitt equation for our constrained dynamical system. We have also considered non-interacting composite systems between the spinorial and reference frame factors. This will be extended further to interacting correlations shortly.

Notice that we have introduced the energy-momentum operator $\widehat{P}_i$ acting on spinorial states such that for all $|K\rangle \in \mathcal{H}_{\text{s}}$

$$\langle K| \widehat{P}_i \widehat{P}^i |\psi_{\text{s}}\rangle := m^2 \psi_{\text{s}}(K). \tag{4.69}$$

To be more precise, taking $|K\rangle$ as the momentum eigenstate for the spin factor of the energy-momentum operator, we have to ensure that such eigenstates are orthogonal, i.e. $\langle K'|K\rangle := \delta(K' - K)$. The condition in Eq.(4.69) ensures that it is a Casimir operator of the Poincaré group which in turn describes mass of the particle. In fact, one has the following momentum eigenstate equation,

$$\langle K| \widehat{P}_i |\psi_{\text{s}}\rangle := K_i \psi_{\text{s}}(K), \tag{4.70}$$

where K_i are the eigenvalues of the energy-momentum operator defined to be the four-wave vector with components $K_i = (\omega_K, -K)$. The temporal component is the Hamiltonian operator $\hat{H}$ with eigenvalues of the angular frequency ω_K for each wave mode K, and the spatial component is given in terms of the momentum $\hat{P}$ with eigenvalues the three-wave vector K for each wave mode. With this information in hand, the four-momentum constraint operator in Eq.(4.67) has components:

$$\begin{aligned}
\widehat{C}_0 : &= \widehat{\text{id}}_{\mathcal{H}_{\text{s}}} \otimes \widehat{p}_0 + \widehat{P}_0 \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}} \\
&= \widehat{\text{id}}_{\mathcal{H}_{\text{s}}} \otimes \widehat{p}_T \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{X}}} + \widehat{H} \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}}. \\
\widehat{C}_\alpha : &= \widehat{\text{id}}_{\mathcal{H}_{\text{s}}} \otimes \widehat{p}_\alpha + \widehat{P}_\alpha \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}} \\
&= \widehat{\text{id}}_{\mathcal{H}_{\text{s}}} \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{T}}} \otimes \widehat{p}_{X^\alpha} + \widehat{P}_\alpha \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}}.
\end{aligned} \tag{4.71}$$

Thus one clearly has a constraint operator similar to the one given via the Page-Wootters construction, except that we have spinorial degrees of freedom and rods and clocks on equal footing, which contrast the former approach. In fact, let us compute the following scalar operator:

$$\begin{aligned}
\widehat{C}_i \widehat{C}^i \left|\Psi_{\text{phy}}\right\rangle &= \left(\widehat{\text{id}}_{\mathcal{H}_{\text{s}}} \otimes \widehat{p}_i + \widehat{P}_i \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}}\right) \left(\widehat{\text{id}}_{\mathcal{H}_{\text{s}}} \otimes \widehat{p}^i + \widehat{P}^i \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}}\right) \left|\Psi_{\text{phy}}\right\rangle \\
&= \left(\widehat{\text{id}}_{\mathcal{H}_{\text{s}}} \otimes \widehat{p}_i \widehat{p}^i + \widehat{P}_i \widehat{P}^i \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}}\right) \left|\Psi_{\text{phy}}\right\rangle + \left(\widehat{P}^i \otimes \widehat{p}_i + \widehat{P}_i \otimes \widehat{p}^i\right) \left|\Psi_{\text{phy}}\right\rangle \\
&= \widehat{C} \left|\Psi_{\text{phy}}\right\rangle + \left(\widehat{P}^i \otimes \widehat{p}_i + \widehat{P}_i \otimes \widehat{p}^i\right) \left|\Psi_{\text{phy}}\right\rangle.
\end{aligned} \tag{4.72}$$

Using Eq.(4.66) and Eq.(4.68), we have $\widehat{C}_i \widehat{C}^i \left|\Psi_{\text{phy}}\right\rangle = 0$. We therefore obtain the mass-shell constraint from the four-momentum constraint operator, provided that the extra constraint operator is satisfied, namely:

$$\left(\widehat{P}^i \otimes \widehat{p}_i + \widehat{P}_i \otimes \widehat{p}^i\right) \left|\Psi_{\text{phy}}\right\rangle = 0. \tag{4.73}$$

Notice that upon applying the reduction map $\left\langle \widehat{\mathcal{R}}_{\text{sys}} \right|$ onto the equation above yields $p^i K_i \Psi(K; p_T, p_X) = 0$ which does indeed mean that the canonical four-momentum and the four-wave vector are orthogonal to each other.

In fact, conditioning the states to the spinorial Hilbert space factor $\mathcal{H}_s$ leads to the following equations:

$$
\begin{aligned}
0 &= \left\langle \widehat{\mathcal{R}}_{\text{sys}} \middle| \widehat{C}_i \left| \Psi_{\text{phy}} \right\rangle \right. \\
&= \left\langle \widehat{\mathcal{R}}_{\text{sys}} \middle| \left(\widehat{\text{id}}_{\mathcal{H}_s} \otimes \widehat{p}_i + \widehat{P}_i \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}} \right) \left| \Psi_{\text{phy}} \right\rangle \right. \\
&= \langle T, X | \, \widehat{p}_i \left| \Psi_{\text{phy}} \right\rangle + \widehat{P}_i \, \langle T, X | \, \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}}) \left| \Psi_{\text{phy}} \right\rangle \\
&= (-i\partial_i) \left| \psi_s(T, X) \right\rangle + \widehat{P}_i \ \left| \psi_s(T, X) \right\rangle,
\end{aligned} \tag{4.74}
$$

or in the much more elegant form

$$
i\partial_i \left| \psi_s(T, X) \right\rangle = \widehat{P}_i \ \left| \psi_s(T, X) \right\rangle, \tag{4.75}
$$

where we used the fact that in the four-position representation of the quantum reference frame, we have the eigenstate equation

$$
\langle T, X | \, \widehat{p}_i \left| \psi_{\text{ext}} \right\rangle := -i\partial_i \psi_{\text{ext}}(T, X). \tag{4.76}
$$

It is surprisingly amazing how Eq.(4.75) looks like a relativistic extension of Schrodinger's equation for which $\partial_i := (\partial_T, \nabla)$. We thus clearly see that by conditioning states to the extended factor we not only have time-evolution but also spacial-evolution in terms of the clock-rode variables, i.e. spacetime-dynamics from constraints upon using the reduction map given by Eq.(4.60).

Spin-0 particle

Consider the case of a spin-0 relativistic particle. The dimension of the spinorial factor is one. We thus can write the Hamiltonian operator for such a system as:

$$
\widehat{H}_0 := \sqrt{\widehat{P} \cdot \widehat{P} + \widehat{P}_i \widehat{P}^i}. \tag{4.77}
$$

Conditioning states to the spinorial factor and using Eq.(4.71) and Eq.(**??**), we get the very familiar results from relativistic quantum mechanics, namely

$$
i\frac{\partial}{\partial T} \left| \psi_s(T, X) \right\rangle = \widehat{H}_0 \left| \psi_s(T, X) \right\rangle, \tag{4.78}
$$

$$
-i\nabla \left| \psi_s(T, X) \right\rangle = \widehat{P} \ \left| \psi_s(T, X) \right\rangle. \tag{4.79}
$$

These are precisely what Schrödinger's time-evolution and three-momentum equations look like for a free relativistic free-particle; and from which the constraint equation given by Eq.(4.66) leads to Klein-Gordon's equation, namely:

$$
(\Box + m^2)\Psi(K; T, X) = 0. \tag{4.80}
$$

We thus get the dynamics of a relativistic spin-0 particle which satisfies Klein-Gordon's equation. Upon solving Eq.(4.78) and Eq.(4.79) by applying the momentum eigenstate $\langle K|$ one has

$$
\Psi(K; T, X) = N e^{-i(\omega_K T - K \cdot X)}, \tag{4.81}
$$

where we used the relation given in Eq.(4.70). This is the wavefunction solution for any mode K over spacetime and is a scalar quantity, i.e. $\Psi(K; T, X) \in \mathbb{C}$. As in usual relativistic-QM negative energy solutions are problematic as they give rise to negative probabilities.

Spin-1/2 particles

Let us now consider the case of a relativistic spin-1/2 particle. The dimension of the spinorial factor is four. We thus have the following Hamiltonian operator:

$$\widehat{H}_{1/2} := \alpha \cdot \widehat{P} + \beta\sqrt{\widehat{P}_i\widehat{P}^i}, \tag{4.82}$$

where α and β satisfy the Clifford algebra over spacetime. In fact, one can clearly see that the Minkowski norm of the energy-momentum operator is given by the mapping

$$\eta : \left(\mathbf{\Gamma}, \widehat{\mathbf{P}}\right) \mapsto \left\|\widehat{\mathbf{P}}\right\|, \tag{4.83}$$

namely, Minkowski norms are defined upon contractions of the Dirac gamma matrices with the vectorial quantity in question; and where the Dirac gamma matrices are given in components by $\mathbf{\Gamma} = (\beta, \beta\alpha)$ for which the algebra is *Clifford's algebra* given by $\{\Gamma^i, \Gamma^j\} = 2\eta^{ij}\ \mathbb{I}_4$.

With such a norm and Hamiltonian in hand, conditioning states to the spinorial factor, we get the familiar result

$$i\frac{\partial}{\partial T}\left|\psi_{\mathrm{s}}(T,X)\right\rangle = \widehat{H}_{1/2}\left|\psi_{\mathrm{s}}(T,X)\right\rangle, \tag{4.84}$$

$$-i\nabla\left|\psi_{\mathrm{s}}(T,X)\right\rangle = \widehat{P}\ \left|\psi_{\mathrm{s}}(T,X)\right\rangle. \tag{4.85}$$

These are precisely what Schrödinger-Dirac's time-evolution and three-momentum equations looks like for a free relativistic free-particle. Combining Eq.(4.84) and Eq.(4.85) appropriately leads to Dirac's equation, namely:

$$(i\Gamma^i\partial_i - m)\Psi(K;T,X) = 0. \tag{4.86}$$

We thus get the dynamics of a relativistic spin-1/2 particle which satisfies Dirac's equation. Upon solving Eq.(4.84) and Eq.(4.85) by applying the momentum eigenstate $\langle K|$ one has

$$\Psi(K;T,X) = N(K)e^{-i(\omega_K T - K\cdot X)}, \tag{4.87}$$

where we used the relation given in Eq.(4.70) and that $\omega_K = \alpha \cdot K + \beta m \equiv \pm\sqrt{K\cdot K + m^2}$ as expected. This is the wavefunction solution for any mode K over spacetime and is a spinor with four components. As usual relativistic-QM negative energy solutions are problematic as they give rise to negative probabilities and might be avoided upon considering antiparticles as negative energy particles travelling backwards in time.

The Duffer-Kemmer-Petiau algebra for spin-0 and spin-1 particles

For the case of spin-0, we have shown previously that the Hamiltonian of the system be given as a square root, which is inconvenient and thus a linear Hamiltonian, just like Dirac's theory would be ideal. A work done by Kemmer in [45] showed that it is possible to write a linear Hamiltonian for spin-0 and spin-1 particles. One can see this from a possible extension of the Dirac gamma matrices and hence a modification to the Minkowski norm Eq.(4.83), namely

$$\eta : \left(\boldsymbol{\beta}, \widehat{\mathbf{P}}\right) \mapsto \left\|\widehat{\mathbf{P}}\right\|, \tag{4.88}$$

where the matrices are given in components by $\boldsymbol{\beta} = (\beta, \beta\alpha)$ for which the algebra is not the Clifford algebra but the *Duffin-Kemmer-Petiau (DKP) algebra* , namely $\beta^i\beta^j\beta^k + \beta^k\beta^j\beta^i = \beta^i\delta^{jk} + \beta^k\delta^{ji}$ for $\delta^{ij} = \mathrm{diag}(1, -1, ..., -1)$. The Hamiltonian for such an algebra is thus given in similar for to the Dirac's Hamiltonian in Eq.(4.69) by

$$\widehat{H}_{\mathrm{DKP}} := \alpha \cdot \widehat{P} + \beta\sqrt{\widehat{P}_i\widehat{P}^i}, \tag{4.89}$$

where $\beta = \beta^0$ and $\alpha = -i[\beta^\alpha, \beta]$. We thus end-up with similar equation to Eqs.(4.84)-(4.85), but also to Eq.(4.86) i.e. the DKP-equation

$$(i\beta^i \partial_i - m)\Psi(K; T, X) = 0. \tag{4.90}$$

The solutions are spinorial solutions analogues of Eq.(4.87), yet for spin-0 one has five spinor components and for spin-1 one has ten spinor components. This is due to the fact that the DKP algebra has three irreducible representations of the Lorentz group.

Possible extension to arbitrary spin has been done in the literature but is mainly a change of Minkowski norm that will define how the algebra is defined, namely a general spin theory would have a Minkowski norm given by matrices $\mathbf{\Omega}$ such that the Minkowski norm

$$\eta : \left(\mathbf{\Omega}, \widehat{\mathbf{P}}\right) \mapsto \left\|\widehat{\mathbf{P}}\right\|,$$

is well defined in spacetime. Of course, it should be noted that for real-world applications, one has to consider non-linear systems rather than linear ones defined above. However, for particles like mesons and electrons, linear theories are accurate enough.

4.3.2 Interacting Hilbert space factors

We have never touched upon that precise topic, namely interacting Hilbert space factors. For starters, let us assume that there are no interactions within the factors of the extended Hilbert space -which is often the case- and interactions between the extended and system factors are present. In that case, the four-momentum constraint operator gets modifications, namely and interacting four-momentum operator $\widehat{P}_i^{\text{int}}$ must be introduced:

$$\widehat{C}_i = \widehat{\text{id}}_{\mathcal{H}_{\text{sys}}} \otimes \widehat{p}_i + \widehat{P}_i \otimes \widehat{\text{id}}_{\mathcal{H}_{\text{ext}}} + \widehat{P}_i^{\text{int}}. \tag{4.91}$$

Such a modification would lead into a modification to the right-hand side of Eq.(4.75), namely

$$\begin{aligned}
i\partial_i \left|\psi_{\text{sys}}(T, X)\right\rangle &= \widehat{P}_i \, \left|\psi_{\text{sys}}(T, X)\right\rangle + \left\langle \widehat{\mathcal{R}}_{\text{sys}} \right| \widehat{P}_i^{\text{int}} \left|\Psi_{\text{phy}}\right\rangle \\
&= \widehat{P}_i \, \left|\psi_{\text{sys}}(T, X)\right\rangle + \int dT' d^3X' \left\langle \widehat{\mathcal{R}}_{\text{sys}} \right| \widehat{P}_i^{\text{int}} \left|T', X'\right\rangle \left|\psi_{\text{sys}}(T', X')\right\rangle \\
&= \widehat{P}_i \, \left|\psi_{\text{sys}}(T, X)\right\rangle + \int dT' d^3X' \, \widehat{k}_i(T, X, T', X') \left|\psi_{\text{sys}}(T', X')\right\rangle \\
&= \left(\widehat{P}_i + \widehat{P}_i^{K}\right) \left|\psi_{\text{sys}}(T, X)\right\rangle,
\end{aligned} \tag{4.92}$$

where we defined the following kernel operator:

$$\widehat{k}_i(T, X, T', X') := \left\langle \widehat{\mathcal{R}}_{\text{sys}} \right| \widehat{P}_i^{\text{int}} \left|T', X'\right\rangle, \tag{4.93}$$

such that the integral kernel operator is

$$\widehat{P}_i^{k} \, \left|\psi_{\text{sys}}(T, X)\right\rangle := \int dT' d^3X' \, \widehat{k}_i(T, X, T', X') \left|\psi_{\text{sys}}(T', X')\right\rangle. \tag{4.94}$$

In fact, we usually demand that $\widehat{P}_i^{\text{int}} = (\widehat{H}^{\text{int}}, 0)$, which means that we do not consider any interacting three-momentums, but consider only interacting Hamiltonians. It is clear that this modification to the relativistic Schrödinger equation is non-local in space and time because of the presence of an interaction term, as the kernel operator does not vanish for $T' \neq T$ and $X' \neq X$: in order to be certain that $\left|\psi_{\text{sys}}(T, X)\right\rangle$ is a solution of Eq.(4.92), one has to know the solutions $\left|\psi_{\text{sys}}(T, X)\right\rangle$ for all time and position, i.e. a solution within the vicinity of T and X are not enough to specify if it is a solution or not.

Chapter 5

Conclusions and Discussions

Throughout this dissertation, we have explored problems regarding time within gravity. For this, we have started by reviewing unimodular gravity in Chapter 2. We started off by providing three alternative approaches for unimodular gravity. The first is Einstein's original attempt of setting $\sqrt{-g} = 1$ after the fully covariant equations have been obtained: this choice allows for simpler coordinate systems and thus simplifies computations. The second way was to introduce this restriction on the metric via a constraint equation, i.e. to hold on-shell: this amounts to introducing the cosmological constant as a Lagrange multiplier and thus getting the constancy condition as an integration constant. This approaches drawback is the breaking of the full diffeomorphism group of spacetime transformations and only to have diffeomorphisms that preserve spacetime volume. The governing equation turned out to be Einstein's trace-free equation, which does not contain any cosmological constant. We have shown further that the full spacetime symmetry can be restored within the Henneaux-Teitelboïm (HT)-gravity, where an auxiliary vector density is introduced to get the unimodular condition, constancy of Λ, and Einstein's fully covariant equations. This formulation of gravity amounts to the introduction of the so-called unimodular time and this is identified as a canonically conjugate observable to the cosmological constant. The problem was to identify what this time really does in terms of spacetime creation.

In Chapter 3, we explored the mathematical realm of knot theory, topological quantum field theory pioneered by Witten within Chern-Simons theory in $2+1$-dimensions, and magnetic helicity that appears in magneto-hydrodynamics. What all these had in common has been emphasized within HT-gravity and unimodular time. The most important feature of this theory is that it is topological in the vector density, namely, there are no locally propagating degrees of freedom. This amount to consider other possible actions that are topological, such as the Pontryagin term or axionic electromagnetism, which is exactly what we considered. The axionic nature of the cosmological constant within HT-gravity when introduced and taken seriously for a system with $U(1)$ gauge symmetry, the unimodular time is identical to magnetic helicity. This has been further connected with knot theory and Chern-Simons theory via Witten's work on knot polynomials and has led us to speculate the remarkable feature that spacetime volume is created by the change in the configurations of these magnetic vortices, which are just linked knots, and is what we experience as the flow of time. We also looked at different extensions such as non-abelian gauge symmetry which introduces interactions among helicities of different flavors, and deconstantizing Nature's other constants, which gives rise to sequestration models, only now with helicities as various unimodular times. The main point is that these vortices and flux tubes are all pointing towards a true meaning for time flow and this is the predictable aspect of these modified HT-gravity theories.

In Chapter 4, we took a step back to the problem of time and tried to come up with a general quantum mechanical framework that fits nicely with relativity. This amounted to introducing a factorization of Hilbert spaces: a system and reference frame Hilbert spaces. Quantum dynamics such as the Schrödinger equations arise from restricting the system Hilbert space on the reference frame Hilbert space. Without restriction mapping, one has constraint operators annihilating physical states, which are identical in most cases to Wheeler-de Witt equations. We also studied the spin-0, spin-$1/2$, and spin-1 cases and obtained standard results from the quantum mechanics of relativistic point-particles. The interacting case has also been discussed very briefly.

With all this being said, lots remains to be explored further. A big emphasis should be placed on the fact that the quantum mechanics of relativistic point-particles is disfavored for quantum fields which is much nicer. A fully functioning quantum field theoretic framework has been recently studied in [28] which requires further investigation and needs to find an application within an actual physical situation, such as unimodular gravity, which is the future intent of the author of this thesis, as we believe a strong connection exists between this new formulation of quantum physics and unimodular gravity. Other open questions exist such as the connection between magnetic helicity and abelian Chern-Simons theory, which needs further investigation, which is another future objective of the author. This deep connection is believed to be fruitful not only for pure mathematical interest but also for the true meaning of time and space.

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